

CONJUGACY OF BOREL SUBALGEBRAS OF RESTRICTED LIE ALGEBRAS AND THE ASSOCIATED SOLVABLE ALGEBRAIC GROUPS (I)

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ABSTRACT. Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie algebra defined over an algebraically closed field \mathbb{K} of characteristic $p > 0$, and G be the adjoint group of \mathfrak{g} . A Borel subalgebra (or Borel for short) of \mathfrak{g} is defined as a maximal solvable subalgebra containing a maximal torus of \mathfrak{g} . Generic Borel subalgebras are by definition a class of Borel subalgebras containing so-called generic Cartan subalgebras. In this paper, we first prove that a conjecture Premet proposed in [17] on regular Cartan subalgebras is valid if and only if it is the case when \mathfrak{g} has generic Cartan subalgebras. We further prove that all maximal solvable subalgebras of \mathfrak{g} are Borels whenever $p > \dim \mathfrak{g}$. We finally classify the conjugacy classes of Borel subalgebras of the restricted simple Lie algebras $W(n)$ under G -conjugation when $p > 3$, and present the representatives of these classes. We also describe the closed connected solvable subgroups of G associated with those representative Borel subalgebras.

INTRODUCTION

In the structure of Lie algebras of connected semi-simple algebraic groups (called classical Lie algebras), Borel subalgebras which is by definition the Lie algebras of Borel subgroups are closely related to Weyl groups, maximal tori (coinciding with Cartan subalgebras). These basic structures constitute most important parts of the machinery of classical Lie algebras, play a fundamental role in the structure and representation theory of classical Lie algebras. There, the basic nature of Borel subalgebras is that for a connected semi-simple algebraic group G and $\mathfrak{g} = \text{Lie}(G)$, all Borel subalgebras of \mathfrak{g} are conjugate under G . We can identify Borel subalgebras with maximal solvable subalgebra containing maximal tori when the ground field is an algebraically closed field of characteristic $\neq 2, 3$ (cf. [10, §14.3] and [19, §III.4]). By contrast, the maximal solvable subalgebras containing maximal tori, which will be still called Borel subalgebras in the present paper, of arbitrary restricted Lie algebras in prime characteristic usually are not longer conjugate. Especially, for non-classical restricted simple Lie algebras, there remain unknown how many and what the conjugacy classes of Borel subalgebras are, and which conjugacy classes of Borel subalgebras will play some key role in the representations of the Lie algebras, as the classical cases. Our motivation is to understand more on Borel subalgebras

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of non-classical restricted Lie algebras, and to exploit some possible connection between the theory of Borel subalgebras, representations and others.

Recall that aside from the analogues of the complex simple Lie algebras (called classical Lie algebras) there are usually four additional classes of restricted simple Lie algebras, the so-called restricted Lie algebras of Cartan type, among of which the Jacobson-Witt algebras $W(n)$ will be our topic of the present paper. In his article [18], Premet studied analogy of Weyl groups and of the Chevalley restriction theorem in the complex simple Lie algebras, and the variety of nilpotent elements for $W(n)$. Continuing this study, Bois, Farnsteiner and the author of the present paper developed some general theory of Weyl groups for restricted Lie algebras, and studied the Weyl groups for the other three classes of Cartan type Lie algebras (cf. [2] and [6]). They proposed in [2] the notion "generic tori" which plays the same important role as the maximal tori in classical Lie algebras, associated with which Weyl groups and the Chevalley restriction theorem were obtained, with aid of the isomorphism classification of maximal tori in [4]. Inspired by the work above-mentioned, we begin the study of Borel subalgebras for restricted Lie algebras. The aim of the paper is to develop some general theory on Borel subalgebras and then to present the conjugacy classes of Borel subalgebras of $W(n)$. Meanwhile, generic Borel subalgebras and the associated solvable subgroups for a non-classical restricted Lie algebra could be expected to play some role as important as Borel subalgebras and Borel subgroups in classical cases, which sheds some new light on the further study of modular representations of non-classical Lie algebras.

Our paper is organized as follows. In the first section, we first give the basic observation the all maximal solvable subalgebras of a restricted Lie algebra \mathfrak{g} are Borels whenever $p > \dim \mathfrak{g}$ (Proposition 1.2). Then we show that the stabilizers of Borel subalgebras in the adjoint groups are closed and solvable. The notion of generic Borel subalgebras are proposed here. This class of special Borel subalgebras are expected to share some special properties in connection with nilpotent cones of Lie algebras. In this section, we also study generic Cartan subalgebras (see §1.2 for the definition), proving that all generic Cartan subalgebras are conjugate under the adjoint group of \mathfrak{g} (Theorem 1.6). Consequently, we prove that a conjecture of Premet proposed in [17] on regular Cartan subalgebras of a finite-dimensional restricted Lie algebra is valid if and only if it is the case when the restricted Lie algebra has generic Cartan subalgebras (Corollary 1.8). In Section 2, we recall some basic results on the structure of W_n , specially propose \mathfrak{t}_r -grading for W_n , associated with the representative of the r th conjugacy class of maximal tori, which will be an important tool in the arguments of isomorphism classification of Borel subalgebras of $W(n)$. Section 3 is devoted to establish all standard Borel subspaces and then to prove their being Lie subalgebras with maximal solvableness. In Section 4, we complete the arguments on isomorphism classification of Borel subalgebras of $W(n)$ (Theorem 4.6) when the characteristic p of the ground field \mathbb{K} is bigger than 3. In Section 5, we prove that the solvable groups associated with Borel subalgebras are connected, and further give a precise description of the solvable subgroups associated with the generic Borel subalgebras. In the concluding section,

we propose some future topics in our research [15] (including the same topics for restricted Lie algebras of other Cartan types).

1. GENERIC BOREL SUBALGEBRAS

Throughout, we shall be working over an algebraically closed field \mathbb{K} of odd characteristic $\text{char}(\mathbb{K}) = p > 0$. Unless mentioned otherwise, all vector spaces are assumed to be finite-dimensional. Given a restricted Lie algebra $(\mathfrak{g}, [p])$, we have an adjoint group $G := \text{Aut}_p(\mathfrak{g})^\circ$, the identity component of its restricted automorphism group.

1.1. Borel subalgebras. A maximal solvable subalgebra \mathcal{B} of a restricted Lie algebra $(\mathfrak{g}, [p])$ is called a *Borel subalgebra* if \mathcal{B} contains a maximal torus of \mathfrak{g} in the sense that a torus \mathfrak{t} is by definition an abelian restricted subalgebra consisting of semi-simple elements, i.e. $X \in (X^{[p]})_p$ for all $X \in \mathfrak{t}$, where $(X^{[p]})_p$ stands for the restricted subalgebra generated by $X^{[p]}$ (cf. [21, §2.3]). A Cartan subalgebra is called a regular one if it contains a torus of maximal dimension.

Let us first give a fact.

Lemma 1.1. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over \mathbb{K} . Then every maximal solvable subalgebra of \mathfrak{g} is a restricted subalgebra.*

Proof. Let \mathcal{B} be a given maximal solvable subalgebra of \mathfrak{g} . Note that the restricted subalgebra $\langle \mathcal{B} \rangle_p$ generated by \mathcal{B} is still solvable. By the maximal solvableness of Borel subalgebras, \mathcal{B} coincides with $\langle \mathcal{B} \rangle_p$. Hence \mathcal{B} itself is restricted. \square

For a given restricted Lie algebra $(\mathfrak{g}, [p])$, the center $C(\mathfrak{g})$ can be decomposed into $C(\mathfrak{g}) = C_s \oplus C_n$ with C_s a subalgebra on which $[p]$ is nonsingular (C_s is called a nonsingular p -subalgebra of $C(\mathfrak{g})$), and C_n the subspace consisting of nilpotent elements in $C(\mathfrak{g})$. Thus, every maximal torus of $(\mathfrak{g}, [p])$ contains a nonsingular p -subalgebra C_s of the center $C(\mathfrak{g})$. Note that any Borel subalgebra of \mathfrak{g} is self-normalized, thereby contains the center of \mathfrak{g} . Hence, the natural surjective morphism $\mathfrak{g} \rightarrow \bar{\mathfrak{g}} := \mathfrak{g}/C(\mathfrak{g})$ gives rises to bijective maps from the set {maximal solvable subalgebras of \mathfrak{g} } onto the set {maximal solvable subalgebras of $\bar{\mathfrak{g}}$ }, and from the set {Borel subalgebras of \mathfrak{g} } onto the set {Borel subalgebras of $\bar{\mathfrak{g}}$ } respectively. So we have the following observation.

Proposition 1.2. *Assume $(\mathfrak{g}, [p])$ is a restricted Lie algebra over \mathbb{K} of characteristic $p > \dim \mathfrak{g}$. Then every maximal solvable subalgebra of \mathfrak{g} is a Borel subalgebra.*

Proof. By the arguments before the lemma, we might as well assume that \mathfrak{g} is a centerless Lie algebra in the following. Thus we can identify \mathfrak{g} with the Lie algebra $\text{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$. Then the p -structure $(\cdot)^{[p]}$ of \mathfrak{g} is inherited to the one of $\text{ad} \mathfrak{g}$ which is just the p th power of the adjoint derivations. Hence we can regard \mathfrak{g} as a restricted subalgebra of $\mathfrak{gl}(\mathfrak{g})$. To prove the proposition, we only need to verify that for a given maximal solvable subalgebra \mathcal{B} of \mathfrak{g} , every maximal torus of \mathcal{B} is maximal in \mathfrak{g} . For this, we take a maximal torus \mathfrak{t} of \mathcal{B} , then extend it a maximal torus \mathfrak{t}_0 in \mathfrak{g} .

Consider a canonical triangular decomposition $\mathfrak{gl}(\mathfrak{g}) = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$, with the space of negative root vectors, and the Cartan subalgebra, and the space of positive root vectors respectively such that $\mathfrak{h} \supset \mathfrak{t}_0$. This is to say, under an fixed basis, \mathfrak{g} can be regarded as $\mathfrak{gl}(\ell, \mathbb{K})$ with $\ell := \dim \mathfrak{g}$, which is decomposed into a direct sum of the space \mathfrak{n}^- of strictly lower-triangular matrices, the space \mathfrak{h} of diagonal matrices, and the space \mathfrak{n}^+ of strictly upper-triangular matrices. For the simplicity of technique in arguments, we might as well assume that \mathfrak{t}_0 consists of all r -minors of left-upper-most diagonals with $r = \dim \mathfrak{t}_0$. It's easily seen that for a given maximal torus \mathfrak{h}' of $\mathfrak{gl}(\ell, \mathbb{K})$ containing \mathfrak{t}_0 , there is an invertible linear transformation $g \in \mathrm{GL}(\ell, \mathbb{K})$ fixing \mathfrak{t}_0 such that g changes \mathfrak{h}' to \mathfrak{h} .

Note that \mathcal{B} is a solvable subalgebra of $\mathfrak{gl}(\ell, \mathbb{K})$ with a maximal torus $\mathfrak{t} \subset \mathfrak{t}_0 \subset \mathfrak{h}$. By the arguments in the previous paragraph, we might as well assume that \mathcal{B} is contained in some Borel subalgebra \mathfrak{b}' of $\mathfrak{gl}(\ell, \mathbb{K})$ such that \mathfrak{b}' has a maximal torus \mathfrak{h} . By [9, Theorem D] and basic arguments of linear algebras, under the assumption $p > \ell$ there exists $\sigma \in \mathrm{GL}(\ell, \mathbb{K})$ fixing \mathfrak{h} , such that $\sigma(\mathfrak{b}') =$ the standard Borel subalgebra $\mathfrak{b} := \mathfrak{h} + \mathfrak{n}^+$, thereby $\sigma(\mathcal{B}) \subset \mathfrak{b}$. Then $\sigma(\mathfrak{t}) \subset \sigma(\mathfrak{t}_0) \subset \mathfrak{b}$, which implies

$$[\sigma(\mathfrak{t}_0), \mathfrak{b}] \subset \mathfrak{b}. \quad (1.1)$$

By $[\sigma(\mathfrak{t}_0), \sigma(\mathfrak{g})] \subset \sigma(\mathfrak{g})$ along with (1.1), we have $[\sigma(\mathfrak{t}_0), \sigma(\mathcal{B})] \subset \mathcal{B}$ because $\sigma(\mathcal{B}) = \sigma(\mathfrak{g}) \cap \mathfrak{b}$. Hence $\sigma(\mathfrak{t}_0) \subset \sigma(\mathfrak{b})$ because the maximal solvable subalgebra $\sigma(\mathcal{B})$ is self-normalized in $\sigma(\mathfrak{g})$, which implies that \mathfrak{t}_0 is a maximal torus of \mathfrak{b} . Hence, $\mathfrak{t} = \mathfrak{t}_0$ is a maximal torus of \mathfrak{g} . We complete the proof. \square

Remark 1.3. (1) As discussed in [9, 10.2], 3-dimensional nilpotent Heisenberg algebra has an irreducible representations of dimension p . This gives a solvable subalgebra \mathcal{B} of $\mathfrak{gl}(p, \mathbb{K})$ which is irreducible on the p -dimensional natural module. Hence, \mathcal{B} is not triangulizable. So \mathcal{B} cannot be any Borel subalgebra of $\mathfrak{gl}(p, \mathbb{K})$.

(2) The restriction $p > \dim \mathfrak{g}$ on the characteristic of the ground field for the proposition can be weakened as $p > \dim V$ for a faithful representation V of least dimension.

In the following, we record a basic result on Borel subalgebras of classical Lie algebras which is important for the sequent arguments.

Proposition 1.4. ([10, 14.3, 14.4]) *Assume the characteristic p of \mathbb{K} is not 2, 3, and $\mathfrak{g} = \mathrm{Lie}(G)$ for a connected reductive group G . Then all Borel subalgebras of \mathfrak{g} are conjugate under $\mathrm{Ad}(G)$ -action.*

1.2. Generic elements and Premet conjecture. For a restricted Lie algebra $(\mathfrak{g}, [p])$, denote by $\mu(\mathfrak{g})$ the maximal dimension of all tori $\mathfrak{t} \subset \mathfrak{g}$. Following Premet, we say that \mathfrak{h} is a regular Cartan subalgebra of \mathfrak{g} if this Cartan subalgebra contains a maximal torus of dimension $\mu(\mathfrak{g})$ (cf. [17]). The dimension of any regular Cartan subalgebra coincides with $r(\mathfrak{g})$ the rank of \mathfrak{g} (cf. [16, Theorem 1]). For a given regular Cartan subalgebra \mathfrak{h} with a maximal torus \mathfrak{t} , one has $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_\alpha$, a root-space decomposition of \mathfrak{g} with respect to \mathfrak{h} , where $\Delta(\mathfrak{g}, \mathfrak{h})$ is the corresponding root system. Then, any two regular Cartan subalgebras in \mathfrak{g} can be obtained from

each other by means of a finite number of elementary switchings defined via the root-space decompositions above mentioned (cf. [17, Theorem 1]).

A torus $\mathfrak{t}_{\text{gen}}$ of $\dim \mathfrak{t}_{\text{gen}} = \mu(\mathfrak{g})$ is called generic if $G \cdot \mathfrak{t}_{\text{gen}}$ is a dense subset of $\overline{\mathbf{S}}_{\mathfrak{g}}$, where $\mathbf{S}_{\mathfrak{g}}$ stands for the Zariski closure of all semisimple elements in \mathfrak{g} (cf. [2, §1]). Set $\mathfrak{h}_{\text{gen}} := C_{\mathfrak{g}}(\mathfrak{t}_{\text{gen}})$, the centralizer of $\mathfrak{t}_{\text{gen}}$ in \mathfrak{g} . By [21, Theorem 2.4.1], $\mathfrak{h}_{\text{gen}}$ is a regular Cartan subalgebra of \mathfrak{g} , which is called a generic Cartan subalgebra. A Borel subalgebra \mathcal{B} of \mathfrak{g} is called generic if it contains a generic Cartan subalgebra $\mathfrak{h}_{\text{gen}}$ of \mathfrak{g} .

Recall that $X \in \mathfrak{g}$ is called regular if the Fitting-nilspace $\mathfrak{g}^0(\text{ad}X) := \{v \in \mathfrak{g} \mid \text{ad}X^{m(X,v)}v = 0 \text{ for some positive integer } m(X,v)\}$ has the minimal dimension among all Fitting-nilspaces when X runs over \mathfrak{g} (cf. [16] and [17]). In this case, $\mathfrak{g}^0(\text{ad}X) = C_{\mathfrak{g}}(X_s)$ is a regular Cartan subalgebra for the Jordan-Chevalley-Seligman decomposition $X = X_s + X_n$ where X_s is semisimple and X_n is p -nilpotent, with $[X_s, X_n] = 0$ (cf. [21, Theorem 2.3.5], [16, Lemma 4, Theorems 1 and 2], [6, Lemma 3.1(1)]) and the comments in [6, Page 4190]). Denote by $\text{Reg}(\mathfrak{g})$ the set of regular elements of \mathfrak{g} . Then $\text{Reg}(\mathfrak{g})$ is an open dense subset of \mathfrak{g} . A regular element X in \mathfrak{g} is called generic if $G \cdot \mathfrak{g}^0(\text{ad}X)$ is a dense subset of \mathfrak{g} . By the definitions and [2, Proposition 1.7], we have

Lemma 1.5. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then \mathfrak{g} has generic Borel subalgebras if and only if one of the following conditions satisfies.*

- (1) *There are generic elements in \mathfrak{g} .*
- (2) *There are generic tori in \mathfrak{g} .*
- (3) *There are generic Cartan subalgebras in \mathfrak{g} .*

If $\mathfrak{h}_{\text{gen}}$ is a generic Cartan subalgebra of \mathfrak{g} containing the generic torus $\mathfrak{t}_{\text{gen}}$, then it follows from [6, Lemma 3.2] that there exists an open dense set U in $\mathfrak{t}_{\text{gen}}$ such that all elements of U are generic. Furthermore, we will see in the forthcoming Corollary 1.8 that in such a case, there exists an open dense set V in \mathfrak{g} such that all elements of V are generic.

Theorem 1.6. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra and \mathfrak{g} contain generic elements. Then all generic Cartan subalgebras (resp. generic tori) are conjugate under G .*

Proof. Note that for a generic torus $\mathfrak{t}_{\text{gen}}$, one has $\mathfrak{t}_{\text{gen}}$ consists of all semisimple elements in the generic Cartan subalgebra $\mathfrak{h}_{\text{gen}} := C_{\mathfrak{g}}(\mathfrak{t}_{\text{gen}})$ (cf. [21, 2.4.1 and 2.4.2]). Hence the conjugation of any two generic tori is equivalent to the conjugation of any two generic Cartan subalgebras. We only need to prove the conjugation property for generic Cartan subalgebras. Take a maximal torus \mathfrak{t} of dimension $\mu(\mathfrak{g})$. Let us first recall an irreducible affine algebraic variety $\mathfrak{X}_{\mathfrak{t}}(\mathbb{K})$ introduced in [6, §2] and [7]. For any given commutative \mathbb{K} -algebra R , by base change $\mathfrak{g} \otimes_{\mathbb{K}} R$ naturally becomes a restricted R -Lie algebra. Define $\mathfrak{T}_{\mathfrak{g}}(R)$ to be the set of homomorphism $\mathfrak{t} \otimes_{\mathbb{K}} R \rightarrow \mathfrak{g} \otimes_{\mathbb{K}} R$ of restricted R -Lie algebras that are split injective when considered R -linear maps. The resulting \mathbb{K} -functor $\mathfrak{T}_{\mathfrak{g}}$ from the commutative \mathbb{K} -algebra category to the set category is the scheme of embeddings from \mathfrak{t} to \mathfrak{g} . By [7, (1.4) and (1.6)], $\mathfrak{T}_{\mathfrak{g}}$ is a

smooth affine scheme of dimension $\dim \mathfrak{g} - r(\mathfrak{g})$. The variety $\mathfrak{X}_t(\mathbb{K})$ is by definition the unique irreducible component exactly containing the inclusion τ , of the variety $\mathfrak{T}_g(\mathbb{K})$ of \mathbb{K} -rational points of \mathfrak{T}_g . The adjoint group G of \mathfrak{g} acts naturally on $\mathfrak{T}_g(\mathbb{K})$ via $g \cdot \varphi = g \circ \varphi$ for any $g \in G$ and $\varphi \in \mathfrak{T}_g(\mathbb{K})$.

By the same arguments in [2, Proposition 1.7], we consider for a generic torus $\mathfrak{t}_{\text{gen}}$, the variety

$$\mathfrak{V} := \{(X, \varphi) \in \mathfrak{g} \times \mathfrak{X}_{\mathfrak{t}_{\text{gen}}}(\mathbb{K}) \mid X \in C_{\mathfrak{g}}(\varphi(\mathfrak{t}_{\text{gen}}))\}.$$

By [6, Proposition 1.6], \mathfrak{V} is an irreducible variety. The projection $\pi_{\mathfrak{g}} : \mathfrak{V} \rightarrow \mathfrak{g}$ via $(X, \varphi) \mapsto X$ is a dominant morphism (cf. [16, Lemma 4, Theorems 1 and 2] or [6, Lemma 3.1]). Since $\mathfrak{t}_{\text{gen}}$ is generic, Proposition 1.4 of [2] provides an element $\varphi_{\text{gen}} \in \mathfrak{X}_{\mathfrak{t}_{\text{gen}}}(\mathbb{K})$ with image $\varphi_{\text{gen}}(\mathfrak{t}_{\text{gen}}) = \mathfrak{t}_{\text{gen}}$ and such that $\overline{G \cdot \varphi_{\text{gen}}} = \mathfrak{X}_{\mathfrak{t}_{\text{gen}}}(\mathbb{K})$. Hence $G \cdot \varphi_{\text{gen}}$ is open in $\mathfrak{X}_{\mathfrak{t}_{\text{gen}}}(\mathbb{K})$. Thus we have an open subset in \mathfrak{V} :

$$\mathfrak{D} := (\mathfrak{g} \times G \cdot \varphi_{\text{gen}}) \cap \mathfrak{V}.$$

Set $\mathfrak{h}_{\text{gen}} = C_{\mathfrak{g}}(\mathfrak{t}_{\text{gen}})$. Then $\mathfrak{h}_{\text{gen}}$ is a generic Cartan subalgebra with $\overline{G \cdot \mathfrak{h}_{\text{gen}}} = \mathfrak{g}$. By Chevalley theorem on constructible sets (cf. [8, §3] or [11, §4.4]), $\pi_{\mathfrak{g}}(\mathfrak{D}) = G \cdot \mathfrak{h}_{\text{gen}}$ is constructible. Hence $G \cdot \mathfrak{h}_{\text{gen}}$ contains an open dense subset of \mathfrak{g} (cf. [11, §4.4]). Note that $\text{Reg}(\mathfrak{g})$ is also an open dense subset of \mathfrak{g} , which is G -stable. So we have a fact that for any another generic Cartan subalgebra $\mathfrak{h}'_{\text{gen}}$ of \mathfrak{g} , $(G \cdot \mathfrak{h}_{\text{gen}}) \cap (G \cdot \mathfrak{h}'_{\text{gen}}) \cap \text{Reg}(\mathfrak{g})$ is nonempty. Take H from the above nonempty set. Then there is a unique regular Cartan subalgebra \mathfrak{h} containing H . Hence \mathfrak{h} coincides with $g_1 \cdot \mathfrak{h}_{\text{gen}}$ and $g_2 \cdot \mathfrak{h}'_{\text{gen}}$ for some $g_1, g_2 \in G$. Thus $\mathfrak{h}_{\text{gen}}$ and $\mathfrak{h}'_{\text{gen}}$ are conjugate. We complete the proof. \square

In [17, Conjecture 2], Premet proposed a conjecture:

Conjecture 1.7. (Premet conjecture) For a finite-dimensional restricted Lie algebra \mathfrak{g} , there exists a nonempty Zariski open subset V consisting of regular elements and such that for any $u, v \in V$ the Cartan subalgebras $\mathfrak{g}^0(\text{adu})$ and $\mathfrak{g}^0(\text{adv})$ are conjugate under G .

From the above theorem, we can prove that Premet conjecture holds when and only when \mathfrak{g} contains generic elements.

Corollary 1.8. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, and G be the adjoint group of \mathfrak{g} . Then the following statements are equivalent.*

- (1) *There exists a nonempty Zariski open subset V consisting of regular elements and such that for any $u, v \in V$ the Cartan subalgebras $\mathfrak{g}^0(\text{adu})$ and $\mathfrak{g}^0(\text{adv})$ are conjugate under G .*
- (2) *The Lie algebra \mathfrak{g} has generic elements.*

Proof. (2) \Rightarrow (1): Suppose \mathfrak{g} has generic elements, or to say, \mathfrak{g} has a generic Cartan subalgebra $\mathfrak{h}_{\text{gen}}$ (cf. Lemma 1.5). By the arguments in the proof of Theorem 1.6, $G \cdot \mathfrak{h}_{\text{gen}}$ is a constructible subset of \mathfrak{g} , thereby has a open dense subset U . Denote $V = U \cap \text{Reg}(\mathfrak{g})$. Then V is non-empty and an open dense subset of \mathfrak{g} . For any $v \in V$, $\mathfrak{g}^0(\text{adv})$ is G -conjugate to $\mathfrak{g}^0(\text{adv}_0)$ for $v_0 \in \mathfrak{h}_{\text{gen}} \cap \text{Reg}(\mathfrak{g})$ with $v = g \cdot v_0$ for $g \in G$. From the uniqueness of the regular Cartan subalgebra containing v_0 it

follows that $\mathfrak{g}^0(\text{adv}_0)$ coincides with $\mathfrak{h}_{\text{gen}}$. Thus $\mathfrak{g}^0(\text{adv})$ is also a generic Cartan subalgebra. Thanks to Theorem 1.6, the Cartan subalgebra $\mathfrak{g}^0(\text{adv})$ and $\mathfrak{g}^0(\text{adv}_0)$ are conjugate under G . The statement (1) follows.

(1) \Rightarrow (2): Suppose that V is a nonempty Zariski open subset of \mathfrak{g} consisting of regular elements and such that the regular Cartan subalgebras $\mathfrak{g}^0(\text{adv})$ for all $v \in V$ are conjugate under G . Note that $v \in \mathfrak{g}^0(\text{adv})$. Hence for any given $v \in V$, $\overline{G \cdot \mathfrak{g}^0(\text{adv})} \supset \overline{V} = \mathfrak{g}$. Thus, the Cartan subalgebra $\mathfrak{h} := \mathfrak{g}^0(\text{adv})$ is generic, thereby v is a generic element of \mathfrak{g} . The statement (2) follows. \square

It is well-known that all regular elements in a classical Lie algebra \mathfrak{g} when the characteristic of the ground field is bigger than 3, are generic (cf. [19, Theorem III 4.1]). According to [2, Proposition 3.3], restricted simple Lie algebras of type W, S and H are of having generic elements. Hence Premet conjecture holds for them. However, there is no generic elements in any restricted simple Lie algebras of type K (cf. [2, Theorem 6.6]). So Premet conjecture fails to hold for the contact restricted simple Lie algebras.

In view of Theorem 1.6, we propose the following question.

Question 1.9. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra admitting generic elements. When are all generic Borel subalgebras conjugate under the adjoint group G ?

We will see that for the Jacobson-Witt algebra $W(n)$, all generic Borel subalgebras are conjugate under G (to see Theorem 4.6).

1.3. Let \mathcal{B} be a Borel subalgebra of a restricted Lie algebra \mathfrak{g} which has the adjoint group G . Set $B = \text{Stab}_G(\mathcal{B})$.

Proposition 1.10. *B is a solvable closed subgroup of G .*

Proof. According to [12, I.2.12(5)] (or [11, Proposition 8.2]), B is a closed subgroup of G . The remaining thing is to prove the solvableness of B . Note that the center $C(\mathfrak{g})$ of \mathfrak{g} is stabilized by G , and is contained in any Borel subalgebras. So we might as well assume that \mathfrak{g} is centerless in the following arguments. By the same arguments as in the proof of Proposition 4.14, we might as well assume without loss of generality, that $\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$, $\mathcal{B} = \mathfrak{g} \cap \mathfrak{b}_0$ where \mathfrak{b}_0 is a standard Borel subalgebra of $\mathfrak{gl}(\mathfrak{g})$, and that G is a closed subgroup of $\text{GL}(\mathfrak{g})$. Note that \mathfrak{b}_0 is the Lie algebra of a Borel subgroup B_0 of $\text{GL}(\mathfrak{g})$, which implies that $B_0 = \text{Stab}_{\text{GL}(\mathfrak{g})}(\mathfrak{b}_0)$. Hence

$$B = \text{Stab}_G(\mathcal{B}) \subset \text{Stab}_{\text{GL}(\mathfrak{g})}(\mathcal{B}) = \text{Stab}_{\text{GL}(\mathfrak{g})}(\mathfrak{b}_0 \cap \mathfrak{g}) = \text{Stab}_{\text{GL}(\mathfrak{g})}(\mathfrak{b}_0) = B_0.$$

Thus, B is solvable. The proof is completed. \square

We call the above B the solvable subgroup of G associated with the Borel subalgebra \mathcal{B} . Suppose \mathcal{B}_{gen} is a generic Borel subalgebra of a restricted Lie algebra $(\mathfrak{g}, [p])$ with the associated solvable group $B_{\text{gen}} = \text{Stab}_G(\mathcal{B}_{\text{gen}})$. We here address a question what we can say about the geometry of G/B_{gen} . In the reductive Lie algebra case, all Borel subalgebras are generic (cf. [3, §14.25]), the associated group are Borel subgroups. There are deep geometry properties on G/B_{gen} and connections with representations (cf. [1], [14] and [20, §8.5.7]). We will address more questions in §6.

2. AUTOMORPHISMS AND STANDARD MAXIMAL TORI OF THE RESTRICTED SIMPLE LIE ALGEBRA $W(n)$

From now on, we assume that \mathbb{K} is an algebraically closed field of characteristic $p \geq 3$.

2.1. Set $\mathbb{P} = \{0, 1, \dots, p-1\}$. For an element $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{P}^m$, we denote $|\mathbf{a}| := a_1 + \dots + a_m$. Define a truncated polynomial algebra $A(n)$ to be the quotient of the polynomial ring $\mathbb{K}[T_1, \dots, T_n]$ by the ideal generated by T_1^p, \dots, T_n^p . Set x_i to be the image of T_i in the quotient. Then $A(n) = \sum_{\mathbf{a} \in \mathbb{P}^n} \mathbb{K} \mathbf{x}^{\mathbf{a}}$, where $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \dots x_n^{a_n}$ with $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{P}^n$. We sometimes write $A(n)$ as $\mathbb{K}[x_1, \dots, x_n]$ for emphasizing those indeterminants. The Jacobson-Witt algebra $W(n)$ is defined as the derivation algebra of $A(n)$. This is to say, $W(n)$ consists of all linear transformations D of $A(n)$ satisfying $D(fg) = D(f)g + fD(g)$ for $f, g \in A(n)$. It's easily seen that $W(n)$ is a free $A(n)$ -module of rank n with basis $\partial_i, i = 1, \dots, n$. Here ∂_i is the image of $\frac{\partial}{\partial T_i}$ in the quotient of $\text{Der}(\mathbb{K}[T_1, \dots, T_n])$ by the ideal generated by $T_i^p, i = 1, \dots, n$. Hence

$$\partial_i \mathbf{x}^{\mathbf{a}} = a_i \mathbf{x}^{\mathbf{a} - \epsilon_i} \quad (2.1)$$

where $\epsilon_i = (\delta_{i,1}, \dots, \delta_{i,n}) \in \mathbb{P}^n$, $\delta_{i,j} = 1$ if $i = j$, and $\delta_{i,j} = 0$ otherwise.

Set $\mathfrak{g} = W(n)$. Then \mathfrak{g} is a \mathbb{Z} -graded restricted simple Lie algebra. The \mathbb{Z} -grading of $W(n)$ arises from the one of the truncated polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ with

$$\mathfrak{g} = \sum_{i=-1}^h \mathfrak{g}_{[i]}, [\mathfrak{g}_{[i]}, \mathfrak{g}_{[j]}] \subset \mathfrak{g}_{[i+j]}, \mathfrak{g}_{[i]}^{[p]} \subset \mathfrak{g}_{[pi]}. \quad (2.2)$$

where $h = n(p-1) - 1$, and we set $\mathfrak{g}_{[i]} := 0$ if i is not between -1 and h . Associated with such a grading, one has a filtration

$$\mathfrak{g} = \mathfrak{g}_{-1} \supset \mathfrak{g}_0 \supset \dots \supset \mathfrak{g}_h \supset 0 \quad (2.3)$$

for $\mathfrak{g}_i = \sum_{j=i}^h \mathfrak{g}_{[j]}$, $i = -1, 0, 1, \dots, h$. For more details on $W(n)$, the readers refer to [21, Ch.4].

2.2. **Automorphisms of $W(n)$.** Recall that an automorphism $\varphi \in \text{Aut}(A(n))$ induces an automorphism $\bar{\varphi}$ of $W(n)$ defined via $\bar{\varphi} : D \mapsto \varphi \circ D \circ \varphi^{-1}$. The inducing correspondence gives rise to a group isomorphism from $\text{Aut}(A(n))$ to $\text{Aut}(W(n))$ (cf. [23]). As to the former, an automorphism is determined by the image of the generators $x_i, i = 1, \dots, n$. We have a criterion that an algebra endomorphism ϕ of $A(n)$ is an automorphism if and only if φ stabilizes the unique maximal ideal of $A(n)$ and the determinant $\det((\partial_i \phi(x_j))_{n \times n})$ is invertible in $A(n)$.

Furthermore, $\text{Aut}(W(n)) = \text{Aut}(W(n))_0 \ltimes \text{Aut}(W(n))_1$, where $\text{Aut}(W(n))_0$ consists of all grading-preserving automorphisms, and $\text{Aut}(W(n))_1$ stands for the unipotent radical. The former is isomorphic to $\text{GL}(n, \mathbb{K})$, which consists of invertible linear transformations on the \mathbb{K} -vector space spanned by $\{x_1, \dots, x_n\}$.

Theorem 2.1. ([23]) *Let $\mathfrak{g} = W(n)$ over \mathbb{K} of characteristic $p \geq 3$ (unless $n = 1$ with assumption $p > 3$). The following statements hold.*

- (1) $\text{Aut}(\mathfrak{g})$ coincides with the adjoint group $G = \text{Aut}_p(\mathfrak{g})^\circ$. Hence it is a connected algebraic group.
- (2) G is a semi-direct product $G = G_0 \ltimes U$, where $G_0 \cong GL(n, \mathbb{K})$ consists of those automorphisms preserving the \mathbb{Z} -grading of \mathfrak{g} , and

$$U = \{g \in G; (g - \text{id}_{\mathfrak{g}})(\mathfrak{g}_i) \subset \mathfrak{g}_{i+1}\}.$$

2.3. Conjugacy classes of the maximal tori and the standard maximal tori. According to Demuškin's result [4], we have the following conjugacy results for maximal tori of $W(n)$.

Theorem 2.2. *Let $\mathfrak{g} = W(n)$. Then the following statements hold.*

- (1) *Two maximal tori $\mathfrak{t}, \mathfrak{t}'$ belong to the same G -orbit if and only if $\dim \mathfrak{t} \cap \mathfrak{g}_0 = \dim \mathfrak{t}' \cap \mathfrak{g}_0$.*
- (2) *There are $(n+1)$ conjugacy classes of maximal tori of \mathfrak{g} . Each maximal torus of \mathfrak{g} is conjugate to one of*

$$\mathfrak{t}_r = \sum_{i=1}^n \mathbb{K} z_i \partial_i, \quad r = 0, 1, \dots, n$$

where $z_i = x_i$ for $i = 1, \dots, n-r$, and $z_i = 1 + x_i$ for $i = n-r+1, \dots, n$.

We call these \mathfrak{t}_r the standard tori of $W(n)$.

2.4. \mathbb{Z} -grading associated with \mathfrak{t}_r . Note that the truncated polynomial algebra $A(n)$ can be presented as the quotient algebra $\mathbb{K}[T_1, \dots, T_n]/(T_1^p - 1, \dots, T_n^p - 1)$. Denote the image T_i by y_i in the previous quotients. Then we can write $A(n)$ as $\mathbb{K}[y_1, \dots, y_n]$. Comparing with the notations appearing in §2.1, we actually have $y_i = 1 + x_i$, $i = 1, \dots, n$. More generally, $A(n)$ can be presented as a truncated polynomial

$$\mathbb{K}[z_1, \dots, z_{n-r}; z_{n-r+1}, \dots, z_n]$$

with generator $z_i := x_i, z_j := y_j$, $i = 1, \dots, n-r; j = n-r+1, \dots, n$, and defining relations:

$$[x_i, x_{i'}] = [y_j, y_{j'}] = [x_i, y_j] = x_i^p = y_j^p - 1 = 0. \quad (2.4)$$

Thus, $W(n)$ can be presented, as a vector space,

$$W(n) = \sum_{i=1}^n \sum_{\mathbf{c}(i) \in \mathbb{P}^n} \mathbb{K} \mathbf{z}^{\mathbf{c}(i)} \partial_i, \quad (2.5)$$

where $\mathbf{z}^{\mathbf{c}(i)} = z_1^{c_1} \dots z_n^{c_n}$ with $\mathbf{c}(i) = (c_1, \dots, c_n) \in \mathbb{P}^n$. Associated to the presentation (2.5), there is a \mathbb{Z} -graded structure as below, called $\mathbb{Z}(\mathfrak{t}_r)$ -grading:

$$W(n) = \bigoplus_s W_{[s]}^{(\mathfrak{t}_r)}, \quad \text{with } W_{[s]}^{(\mathfrak{t}_r)} = \mathbb{K}\text{-Span}\{\mathbf{z}^{\mathbf{c}(i)} \partial_i \mid |\mathbf{c}(i)| = s+1, i = 1, \dots, n\}. \quad (2.6)$$

Actually, every homogenous space $W_{[s]}^{(\mathfrak{t}_r)}$ is a \mathfrak{t}_r -module. However, $W(n)$ may not be a graded algebra, associated to such a graded structure unless $r = 0$. For the

case of $r = 0$, the associated graded structure in (2.6) is called a standard-graded structure, coinciding with the one in (2.2). It's worthwhile mentioning a fact that v is an s -graded homogenous element of $W(n)$ if and only if

$$\text{ad}T_r(v) = sv \quad (2.7)$$

with $T_r := \sum_{i=1}^n z_i \partial_i$, where $z_i = x_i$ and $z_j = 1 + x_j$, $i = 1, \dots, n-r$; $j = n-r+1, \dots, n$.

When talking about \mathfrak{t}_0 , we will omit the superscript for the associated graded structure as below

$$W(n) = \bigoplus_s W_{[s]}, \text{ with } W_{[s]} = \mathbb{K}\text{-Span}\{\mathbf{x}^{\mathbf{c}(i)} \partial_i \mid |\mathbf{c}(i)| = s+1, i = 1, \dots, n\}. \quad (2.8)$$

This gives rise to a \mathbb{Z} -graded Lie algebra structure for $W(n)$ as shown as in (2.2). Thanks to Theorem 2.1, the associated filtration is invariant under $\text{Aut}(W(n))$.

2.5. Let \mathfrak{H} be a subalgebra of $W(n)$. Call \mathfrak{H} a $\mathbb{Z}(\mathfrak{t}_r)$ -graded subalgebra if $\mathfrak{H} = \sum_i \mathfrak{H}_{[i]}$ with $\mathfrak{H}_{[i]} = \mathfrak{H} \cap W(n)_{[i]}^{(\mathfrak{t}_r)}$.

Lemma 2.3. (1) *A subalgebra containing \mathfrak{t}_0 must be standard-graded.*
 (2) *A subalgebra containing \mathfrak{t}_r must be $\mathbb{Z}(\mathfrak{t}_r)$ -graded.*

Proof. The first assertion is a special case of the second one. So we only need prove the second one. Let \mathfrak{H} be a subalgebra of $W(n)$ containing \mathfrak{t}_r . Then \mathfrak{H} be a submodule of $W(n)$, as a \mathfrak{t}_r -module. There is a direct decomposition of weight spaces $\mathfrak{H} = C_{\mathfrak{H}}(\mathfrak{t}_r) + \sum_{\alpha \in \mathfrak{t}_r^* \setminus \{0\}} \mathfrak{H}_{\alpha}$, with $\mathfrak{H}_{\alpha} = \{v \in \mathfrak{H} \mid \text{ad}H(v) = \alpha(H)v \text{ for all } H \in \mathfrak{t}_r\}$. Take a basis consisting of the toral elements $\{H_i := z_i \partial_i \mid i = 1, \dots, n\}$ of \mathfrak{t}_r , this is to say $H_i^{[p]} = H_i$, $i = 1, \dots, n$. Then $\alpha \in \mathfrak{H}^*$ is determined by the values $\alpha_i := \alpha(H_i)$, $i = 1, \dots, n$. From the fact that \mathfrak{H} is a restricted \mathfrak{t}_r -module, it follows that $\alpha_i \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. So we can regard α as an element of \mathbb{F}_p^n . Hence we identify \mathfrak{t}_r^* with \mathbb{F}_p^n . For $v \in \mathfrak{H}$, we can write $v = v_0 + \sum_{\alpha} v_{\alpha}$ with $v_0 \in C_{\mathfrak{H}}(\mathfrak{t}_r)$, and $v_{\alpha} \in \mathfrak{H}_{\alpha}$. Let us investigate v_0 and all summands v_{α} of weight α in the following.

Thanks to (2.6), we can write $v_0 = \sum_q v_{0,q}$ for $v_{0,q} \in W_{[q]}^{(\mathfrak{t}_r)}$. Further, we write $v_{0,q}$ as below, following (2.5):

$$v_{0,q} = \sum C_{\mathbf{c}(i)} \mathbf{z}^{\mathbf{c}(i)} \partial_i \quad (2.9)$$

where all $C_{\mathbf{c}(i)} \in \mathbb{K}$, $\mathbf{c}(i) = (\mathbf{c}(i)_1, \dots, \mathbf{c}(i)_n) \in \mathbb{P}^n$. Put $z_s \partial_s \in \mathfrak{t}_r$ to act on the both sides of (2.9), we then have

$$[z_s \partial_s, v_{0,s}] = 0 = (\mathbf{c}(s)_s - 1) C_{\mathbf{c}(s)} \mathbf{z}^{\mathbf{c}(s)} \partial_s + \sum_{i \neq s} \mathbf{c}(i)_s C_{\mathbf{c}(i)} \mathbf{z}^{\mathbf{c}(i)} \partial_i.$$

When s runs over $\{1, \dots, n\}$, we have $\mathbf{z}^{\mathbf{c}(i)} \partial_i = z_i \partial_i$ as long as $C_{\mathbf{c}(i)} \neq 0$. This is to say $v_{0,q} \in \sum \mathbb{K}_{i=1}^n z_i \partial_i$. Thus, all $v_{0,q}$, thereby v_0 , actually fall in \mathfrak{t}_r . Acturally, we have proved that $C_{\mathfrak{H}}(\mathfrak{t}_r)$ coincides with \mathfrak{t}_r .

Next, let us investigate v_α for $\alpha \in \mathfrak{t}_r^* \setminus \{0\}$. Similarly to the above, we write $v_\alpha = \sum_q v_{\alpha,q}$ for $v_{\alpha,q} \in W_{[q]}^{(\mathfrak{t}_r)}$. To complete the proof of the lemma, it suffices to prove the following statement

$$\text{all } v_{\alpha,q} \text{ fall in } \mathfrak{H}. \quad (2.10)$$

For this, we write $v_{\alpha,q}$ as below, following (2.5):

$$v_{\alpha,q} = \sum C_{\mathbf{d}(i)} \mathbf{z}^{\mathbf{d}(i)} \partial_i \quad (2.11)$$

where all $C_{\mathbf{d}(i)} \in \mathbb{K}$, $\mathbf{d}(i) = (\mathbf{d}(i)_1, \dots, \mathbf{d}(i)_n) \in \mathbb{P}^n$. Similarly, put $H_s = z_s \partial_s \in \mathfrak{t}_r$ to act on the both sides of (2.11), we then have

$$[H_s, v_{\alpha,q}] = \alpha_s v_{\alpha,q} = (\mathbf{d}(s)_s - 1) C_{\mathbf{d}(s)} \mathbf{z}^{\mathbf{d}(s)} \partial_s + \sum_{i \neq s} \mathbf{d}(i)_s C_{\mathbf{d}(i)} \mathbf{z}^{\mathbf{d}(i)} \partial_i. \quad (2.12)$$

We can naturally regard α_s as an element of \mathbb{P} , in a unique way. Then Equation (2.12) implies $\mathbf{d}(i)_s = \alpha_s$ for $i \neq s$ provided $C_{\mathbf{d}(i)} \neq 0$, and $\mathbf{d}(s)_s = \alpha_s + 1$ provided $C_{\mathbf{d}(s)} \neq 0$. When s runs over $\{1, \dots, n\}$, we have $\mathbf{z}^{\mathbf{d}(i)} \partial_i = z_i^{\alpha_i+1} \prod_{j \neq i} z_j^{\alpha_j} \partial_i$ as long as $C_{\mathbf{d}(i)} \neq 0$. This means q is unique, which is equal to $\alpha_1 + \dots + \alpha_n \in \mathbb{Z}_{\geq 0}$ whenever we regard α uniquely as an element of \mathbb{P}^n . Hence $v_\alpha = v_{\alpha,q}$. Thus Statement (2.10) is true. Actually, we have proved further that $v_\alpha \in \mathfrak{H}_\alpha$ is just a homogeneous $\mathbb{Z}(\mathfrak{t}_r)$ -graded element of degree $\alpha_1 + \dots + \alpha_n$. Thus we have proved that any $v \in \mathfrak{H}$ with the grading expression $v = \sum_i v_i$ as in (2.6), one has $v_i \in \mathfrak{H}$. We complete the proof. \square

According to the arguments in the above proof, we can refine $\mathbb{Z}(\mathfrak{t}_r)$ -grading. When taking a toral basis $\{H_i := z_i \partial_i \mid i = 1, \dots, n\}$ of \mathfrak{t}_r , we can regard $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathfrak{t}_r^*$ as an element of \mathbb{F}_p^n . Hence we identify \mathfrak{t}_r^* with \mathbb{F}_p^n . The above proof shows that if \mathfrak{H} contains \mathfrak{t}_r , then $\mathfrak{H} = \sum_{\alpha \in \mathbb{P}^n} \mathfrak{H}_\alpha$, with $\mathfrak{H}_\alpha = \{v \in \mathfrak{H} \mid \text{ad} H_i(v) = \alpha_i v\}$. Here we naturally regard $\alpha_i \in \mathbb{F}_p$ as an element of \mathbb{P} , in a unique way. In this sense, we call \mathfrak{H} having \mathfrak{t}_r -grading. Furthermore, $\mathfrak{H} = \sum_i \mathfrak{H}_{[i]}$ with $\mathfrak{H}_{[i]} = \sum_{\alpha \in \mathbb{P}^n, \alpha_1 + \dots + \alpha_n = i} \mathfrak{H}_\alpha$. Summing up, we have

Corollary 2.4. (1) *A subalgebra containing \mathfrak{t}_0 , must be \mathfrak{t}_0 -graded.*

(2) *A subalgebra containing \mathfrak{t}_r , must be \mathfrak{t}_r -graded.*

Remark 2.5. From the arguments in the proof of Lemma 2.3, we know that $\text{Cent}_{\mathfrak{g}}(\mathfrak{t}_r) = \mathfrak{t}_r$ for $\mathfrak{g} = W(n)$ and all $r = 0, 1, \dots, n$. By Theorem 2.2 and [21, Theorem 2.4.1], the maximal tori of $W(n)$ coincides with the Cartan subalgebras.

2.6. Recall $W(n)_{[0]} \cong \mathfrak{gl}(n, \mathbb{K})$ under the mapping $x_i \partial_j \mapsto E_{ij}$, where E_{ij} means the $n \times n$ matrix with all entries being zero unless (i, j) -entry whose value is 1. We then have a triangular decomposition $W(n)_{[0]} = \sum_{i < j} \mathbb{K} x_j \partial_i + \mathfrak{t}_0 + \sum_{i < j} \mathbb{K} x_i \partial_j$. Denote $\mathfrak{b} = \mathfrak{t}_0 + \sum_{i < j} \mathbb{K} x_i \partial_j$, which is a standard Borel subalgebra of $W(n)_{[0]}$. We identify $W(n)_{[0]}$ with $\mathfrak{gl}(n, \mathbb{K})$, and $\text{Aut}(W(n))_0$ with $\text{GL}(n, \mathbb{K})$ in the sequent arguments whenever the context is clear.

Lemma 2.6. *Assume the characteristic p of the ground field \mathbb{K} is bigger than 3, and $W(n)_0 = \sum_{i \geq 0} W(n)_{[i]}$. The following statements hold.*

- (1) All maximal tori in $W(n)_0$ are conjugate to \mathfrak{t}_0 under $\text{Aut}(W(n))$.
- (2) All Borel subalgebras of $W(n)_0$ are conjugate to $\mathcal{B}_0 := \mathfrak{b} + W(n)_1$ under $\text{Aut}(W(n))$.

Proof. (1) Set $\mathfrak{g} = W(n)$ and $G = \text{Aut}(W(n))$. Denote the standard-graded and filtered structure by $\mathfrak{g} = \sum_i \mathfrak{g}_{[i]}$ and $\{\mathfrak{g}_i\}$ respectively. According to Demškin's result (cf. [4]), we only need to show that maximal tori of \mathfrak{g}_0 are also the ones of \mathfrak{g} . Suppose that \mathfrak{t} is a maximal torus of \mathfrak{g}_0 with basis $Z_i, i = 1, \dots, m$. We can write $Z_i = T_i + V_i$ with $T_i \in \mathfrak{g}_{[0]}$ and $V_i \in \mathfrak{g}_1$. Note that V_i is nilpotent. Hence T_i must be a semisimple element. Thus the set $\{T_i, i = 1, \dots, m\}$ spans a maximal torus of \mathfrak{g}_0 , thereby the maximal torus of $\mathfrak{g}_{[0]}$. The latter is conjugate to \mathfrak{t}_0 under $\text{GL}(n, \mathbb{K})$ (cf. [19, Theorem III.4.1]). Hence the first assertion is proved.

(2) We first observe that for any Borel subalgebra \mathfrak{b}' of $\mathfrak{g}_{[0]} \cong \mathfrak{gl}(n, \mathbb{K})$, $\mathcal{B}' := \mathfrak{b}' + \mathfrak{g}_1$ must be a Borel subalgebra of \mathfrak{g}_0 . The solvableness of \mathcal{B}' comes from the fact that $\mathcal{B}'^{(n)} \subset \mathfrak{g}_1$, and \mathfrak{g}_1 is nilpotent, where $L^{(m)}$ for a Lie algebra L is defined as $[L^{(m-1)}, L^{(m-1)}]$ by induction with $L^{(0)} = L$. As to the maximal solvableness of \mathcal{B}' , observe that $\mathcal{B}' \subset \mathfrak{g}_0$ is standard-graded with $\mathcal{B}'_{[0]} = \mathfrak{b}'$ and $\mathcal{B}'_{[i]} = \mathfrak{g}_{[i]}$ for all $i > 0$. So any other solvable algebra of \mathfrak{g}_0 containing \mathcal{B}' is also standard-graded. Then the maximal solvableness of \mathfrak{b}' in $\mathfrak{g}_{[0]}$ implies that of \mathcal{B}' in \mathfrak{g}_0 . Note that any Borel subalgebra of $\mathfrak{gl}(n, \mathbb{K})$ are conjugate to the standard one \mathfrak{b} under $\text{GL}(n, \mathbb{K}) \subset \text{Aut}(W(n))$ (cf. Proposition 1.4). It follows that any Borel subalgebra of \mathfrak{g}_0 up to conjugate, can be constructed in this way.

By the invariance of the filtration $\{\mathfrak{g}_i\}$ under G , we know those Borel subalgebras are conjugate to $\mathcal{B}_0 = \mathfrak{b} + W(n)_1$ under $\text{GL}(n, \mathbb{K})$. We complete the proof. \square

2.7. An application to $W(1)$. We assume $\mathfrak{g} = W(1)$ and the characteristic p of the ground field is bigger than 3. In this special case, we adopt the notations x and ∂ , with the same meaning as x_1 , and ∂_1 respectively. By Lemma 2.3, we can list the conjugation result of Borel subalgebras in $W(1)$ as below.

Proposition 2.7. *There are only two conjugacy classes of the Borel subalgebras for the Witt algebra $W(1)$. The standard representatives are $\mathbb{K}\partial + \mathbb{K}x\partial$ and $\mathbb{K}x\partial + \mathbb{K}x^2\partial + \dots + \mathbb{K}x^{p-1}\partial$.*

Proof. Recall $W(1)$ admits two conjugacy classes of maximal tori, with representatives $\mathfrak{t}_0 = \mathbb{K}x\partial$, and $\mathfrak{t}_1 = \mathbb{K}(1+x)\partial$. For a given Borel subalgebra \mathcal{B} , we can assume that up to conjugation, either $\mathcal{B} \supset \mathfrak{t}_0 = \mathbb{K}x\partial$, or $\mathcal{B} \supset \mathfrak{t}_1 := \mathbb{K}(1+x)\partial$.

In the case when $\mathcal{B} \supset \mathfrak{t}_0$, Corollary 2.4 implies that \mathcal{B} is standard-graded. So $\mathcal{B} = \sum_i \mathcal{B}_{[i]}$, where $\mathcal{B}_{[0]} = W_{[0]}$. If $\dim \mathcal{B}_{[-1]} = 1$, then the maximal solvable subalgebra $\mathcal{B} \supset \mathbb{K}\partial + \mathbb{K}x\partial$. However, the latter is already a maximal solvable subalgebra. So $\mathcal{B} = \mathbb{K}\partial + \mathbb{K}x\partial$ in this case. If $\dim \mathcal{B}_{[-1]} = 0$, then $\mathcal{B} \subset W(1)_0$. The latter is solvable. The maximal solvableness of \mathcal{B} implies $\mathcal{B} = W(1)_0$. In the following argument, we assume that $\mathcal{B} \supset \mathfrak{t}_1$.

According to Lemma 2.3, \mathcal{B} is \mathfrak{t}_1 -graded. The $\mathbb{Z}(\mathfrak{t}_1)$ -graded structure is denoted by $W(1) = \bigoplus_i W(1)_{[i]}^{(\mathfrak{t}_1)}$. In the case $\dim \mathcal{B}_{[-1]}^{(\mathfrak{t}_1)} = 1$, $\mathcal{B} \supset \mathbb{K}\partial + \mathbb{K}(1+x)\partial = \mathbb{K}\partial + \mathbb{K}x\partial = \mathcal{B}_1$. Hence $\mathcal{B} = \mathcal{B}_1$. So we only need to consider the situation that

$\mathcal{B} \supset \mathfrak{t}_1$ and $\dim \mathcal{B}_{[-1]}^{(\mathfrak{t}_1)} = 0$. In such a case, $\mathcal{B} \subset \mathbb{K}(1+x)\partial + \sum_{i>0} \mathbb{K}(1+x)^i\partial$. As \mathcal{B} is a maximal solvable subalgebra, $\mathcal{B} \supsetneq \mathbb{K}(1+x)\partial$. Therefore, there exists $D := (1+x)^a\partial \in \mathcal{B}$ with $p > a > 1$, which implies that \mathcal{B} coincides with $\mathbb{K}(1+x)\partial + \mathbb{K}(1+x)^a\partial$ because the latter is a maximal solvable subalgebra. Define $\varphi \in \text{Aut}(W(1))$ via $\varphi(x) = (1+x)^b - 1$, where $b \in \{2, \dots, p-1\}$ with $ab \equiv 1 \pmod{p}$. Then the inverse $\varphi^{-1} : x \mapsto (1+x)^{p-a+1} - 1$. We have an automorphism $\bar{\varphi}$ of $W(1)$ induced by φ , denoted by Φ . By a straightforward computation, we have

$$\begin{aligned}\Phi((1+x)^a\partial) &= (p-a+1)\partial, \\ \Phi((1+x)\partial) &= (p-a+1)(1+x)\partial.\end{aligned}\tag{2.13}$$

Under the transformation via such a Φ , we have $\mathcal{B} \cong \mathbb{K}\partial + \mathbb{K}(1+x)\partial$. \square

Remark 2.8. For $\mathfrak{g} = W(1)$, we can show that any maximal solvable subalgebra is a Borel. Actually, for a given maximal subalgebra \mathcal{B} , it can be endowed with a filtration structure $\{\mathcal{B}_i\}$ inheriting the one of \mathfrak{g} as in (2.3). Consider the graded subalgebra $\text{Gr}(\mathcal{B})$ of \mathfrak{g} . Then $\text{Gr}(\mathcal{B}) = \sum_{i=-1}^{p-2} \text{Gr}(\mathcal{B})_{[i]}$ is also a maximal solvable subalgebra of \mathfrak{g} . Now $\text{Gr}(\mathcal{B})$ is normalized by $\mathbb{K}x\partial$. Hence, $\text{Gr}(\mathcal{B})$ contains $\mathbb{K}x\partial$. This implies that \mathcal{B} contains nonzero semi-simple elements, and then contains some maximal torus of \mathfrak{g} . Therefore, the above proposition covers the main result of [24].

3. STANDARD BOREL SUBALGEBRAS

We maintain the notations as before.

3.1. Borel spaces. We will first introduce $(n+1)$ vector subspaces \mathcal{B}_q in $W(n)$, $q = 0, 1, \dots, n$. Recall $\mathcal{B}_0 = \mathfrak{b} + W(n)_1$, where $W_1 = \sum_{i \geq 1} W(n)_{[i]}$. Next we set

$$\mathcal{B}_n = W(n)_{[-1]} + \mathfrak{b} + \sum_{q=1}^n \sum_{\mathbf{a}(q)} \mathbb{K} \mathbf{x}^{\mathbf{a}(q)} \partial_q,$$

where $\mathbf{a}(q) := (a_1, \dots, a_q) \in \mathbb{P}^q$ with $|\mathbf{a}(q)| > 1$, a_i runs over \mathbb{P} for $i = 1, \dots, q-1$, and $a_q = 0, 1$. We call \mathcal{B}_0 and \mathcal{B}_n the nought-varied Borel space, and the full-varied Borel space respectively.

Before giving general Borel spaces \mathcal{B}_q , we make an appointment of some conventions as below (some have been appeared before, but now presented formally).

Conventions 3.1. Let z_i be either x_i or $(1+x_i)$, $i = 1, \dots, n$. For a subsequence $\mathbf{u} = (u_1, \dots, u_q)$ of the sequence (z_1, \dots, z_n) i.e., $\{u_1, \dots, u_q\} \subset \{z_1, \dots, z_n\}$ we adopt the notations

- (1) Set $\mathcal{B}_0(u_1, \dots, u_q)$ and $\mathcal{B}_q(u_1, \dots, u_q)$ to be the nought-varied Borel subspace \mathcal{B}_0 , and the full-varied Borel subspace \mathcal{B}_q of $W(q)$ respectively. Here $W(q)$ is the derivation algebra of $A(q) = \mathbb{K}[u_1, \dots, u_q]$, which is the subalgebra of the truncated polynomial algebra $A(n) = \mathbb{K}[x_1, \dots, x_n]$ generated by u_1, \dots, u_q . For distinguishing indeterminants, we sometimes need the notations $A(u_1, \dots, u_q)$ and $W(u_1, \dots, u_q)$ standing for those $A(q)$ and $W(q)$ respectively.

- (2) Set $\mathbf{t}_0(u_1, \dots, u_q) := x_{i_1} \partial_{i_1} + \dots + x_{i_q} \partial_{i_q}$, and $\mathbf{t}_q(u_1, \dots, u_q) = (1 + x_{i_1}) \partial_{i_1} + \dots + (1 + x_{i_q}) \partial_{i_q}$ when $(u_1, \dots, u_q) = (x_{i_1}, \dots, x_{i_q})$. Those \mathbf{t}_0 and \mathbf{t}_q mean the first and last standard tori of $W(x_{i_1}, \dots, x_{i_q})$ respectively.
- (3) Set $\mathbf{u}^{\mathbf{a}} := u_1^{a_1} \dots u_q^{a_q}$ if $\mathbf{a} = (a_1, \dots, a_q) \in \mathbb{P}^q$.

It's the position for us to define the q th Borel space \mathbf{B}_q . Let us first take $\mathbf{u} = (x_1, \dots, x_{n-q})$ and $\mathbf{w} = (x_{n-q+1}, \dots, x_n)$. Define

$$\mathbf{B}_q = \mathbf{B}_0(x_1, \dots, x_{n-q}) + Q_q + \mathbf{B}_q(x_{n-q+1}, \dots, x_n),$$

where

$$Q_q = \sum_{i=1}^{n-q} \sum_{\mathbf{a}(i), \mathbf{b}(i)} \mathbb{K} \mathbf{u}^{\mathbf{a}(i)} \mathbf{w}^{\mathbf{b}(i)} \partial_i + \sum_{j=n-q+1}^n \sum_{\mathbf{a}(j), \mathbf{b}(j)} \mathbb{K} \mathbf{u}^{\mathbf{a}(j)} \mathbf{w}^{\mathbf{b}(j)} \partial_j, \quad (3.1)$$

where $\mathbf{a}(i) := (a_1, \dots, a_{n-q}) \in \mathbb{P}^{n-q}$ is subjected to the condition that either $|\mathbf{a}(i)| > 1$ or $|\mathbf{a}(i)| = 1 = a_1 + \dots + a_i$, while $\mathbf{a}(j) \in \mathbb{P}^{n-q}$ is subjected to the condition $|\mathbf{a}(j)| > 0$, and $\mathbf{b}(-) := (b_{n-q+1}, \dots, b_n)$ runs over \mathbb{P}^r for $(-) = (i), (j)$, subjected to the condition $|\mathbf{b}(i)| > 0$.

In the sequent subsections, we will prove that all \mathbf{B}_q are Borel subalgebras of $W(n)$.

3.2. Varied positive root systems of the rigid root system. Set $E = \{-1, 0\}$. Consider a subset Δ of $\Gamma := \mathbb{P}^n \times E^n$:

$$\Delta = \{(a_1, \dots, a_n) \times (\eta_1, \dots, \eta_n) \in \Gamma \mid \eta_1 + \dots + \eta_n = -1\},$$

and then set

$$\bar{\Delta} = \Delta \cup \{\infty\}.$$

For $\alpha = (\sum_i a_i \epsilon_i) \times (-\epsilon_s)$ and $\beta = (\sum_i b_i \epsilon_i) \times (-\epsilon_t) \in \Delta$, we define an ordered operator " $+$ ": $\bar{\Delta} \times \bar{\Delta} \rightarrow \bar{\Delta}$ as $\alpha + \beta := (\sum_i (a_i + b_i) \epsilon_i - \epsilon_s) \times (-\epsilon_t)$ if it lies in Δ , and $\alpha + \beta := \infty$ otherwise; and define $\infty + \text{any} = \text{any} + \infty = \infty$. Then $W(n)$ can be decomposed into a direct sum of one-dimensional root spaces, associated with the so-called rigid root system as below:

$$W(n) = \sum_{\alpha \in \bar{\Delta}} W(n)_\alpha$$

where $W(n)_\alpha := \mathbb{K} x_1^{a_1} \dots x_n^{a_n} \partial_j$ for $\alpha = (a_1, \dots, a_n) \times (-\epsilon_j)$, and $W(n)_\infty := 0$. We call $\bar{\Delta}$ the rigid root system of $W(n)$. Then we have

$$[W(n)_\alpha, W(n)_\beta] \subset W(n)_{\alpha+\beta} + W(n)_{\beta+\alpha}. \quad (3.2)$$

Associated with the naught-varied, and full-varied Borel subspaces, we define positive root systems $\bar{\Delta}(0)_+$, and $\bar{\Delta}(n)_+$ respectively as below

$$\begin{aligned}\bar{\Delta}(0)_+ &:= \{\epsilon_i \times (-\epsilon_j) \mid 1 \leq i \leq j \leq n\} \cup \{\infty\} \\ &\cup \left\{ \sum_{i=1}^n a_i \epsilon_i \times (-\epsilon_j) \mid a_1 + \cdots + a_n > 1, j = 1, \dots, n \right\}; \\ \bar{\Delta}(n)_+ &:= \{0 \times (-\epsilon_j) \mid j = 1, \dots, n\} \cup \{\epsilon_i \times (-\epsilon_j) \mid 1 \leq i \leq j \leq n\} \cup \{\infty\} \\ &\cup \left\{ \sum_{i=1}^n a_i \epsilon_i \times (-\epsilon_j) \mid a_1 + \cdots + a_n = a_1 + \cdots + a_j > 1; a_j = 0, 1; j = 1, 2, \dots, n \right\}.\end{aligned}$$

Then $B_0 = \sum_{\alpha \in \bar{\Delta}(0)_+} W(n)_\alpha$ and $B_n = \sum_{\alpha \in \bar{\Delta}(n)_+} W(n)_\alpha$. Generally, we set

$$\bar{\Delta}(q)_+ = \bar{\Delta}^{(x_1, \dots, x_{n-q})}(0)_+ \cup \bar{\Delta}^{(x_{n-q+1}, \dots, x_n)}(q)_+ \cup \bar{\Delta}\{Q\}_+,$$

where $\bar{\Delta}^{(x_1, \dots, x_{n-q})}(0)_+$, and $\bar{\Delta}^{(x_{n-q+1}, \dots, x_n)}(q)_+$ denote the nought-varied positive root system and full-varied positive root system associated with $\mathbb{K}[x_1, \dots, x_{n-q}]$ and $\mathbb{K}[x_{n-q+1}, \dots, x_n]$ respectively; and $\bar{\Delta}\{Q\}_+$ denotes a subset of $\bar{\Delta}_+$ as below:

$$\begin{aligned}\bar{\Delta}\{Q\}_+ &= \left\{ \left(\sum_{i=1}^{n-q} a_i \epsilon_i + \sum_{j=n-q+1}^n b_j \epsilon_j \right) \times (-\epsilon_r) \mid 1 \leq r \leq n-q, \text{ either } a_1 + \cdots + a_{n-q} > 1 \right. \\ &\quad \left. \text{or } a_1 + \cdots + a_{n-q} = a_1 + \cdots + a_r = 1; b_{n-q+1} + \cdots + b_n > 0 \right\} \cup \{\infty\} \cup \\ &\quad \left\{ \left(\sum_{i=1}^{n-q} a_i \epsilon_i + \sum_{j=n-q+1}^n b_j \epsilon_j \right) \times (-\epsilon_s) \mid n-q+1 \leq s \leq n, a_1 + \cdots + a_{n-q} > 0 \right\}.\end{aligned}$$

By the above construction, we have

Lemma 3.2. *Maintain the notations as above. The following statements hold.*

- (1) $B_q = \sum_{\alpha \in \bar{\Delta}(q)_+} W(n)_\alpha$, $q = 0, 1, \dots, n$.
- (2) B_q contains the maximal tori \mathfrak{t}_r , $r = 0, 1, \dots, q$.
- (3) Furthermore, B_q is a restricted \mathfrak{t}_r -module for $r = 0, 1, \dots, q$. As a \mathfrak{t}_0 -module the weight space is below, in the sense of the argument in the proof of Lemma 2.3:

$$\{\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{F}_p^n \mid a_i \text{ coincides with the } i\text{th entry of } \bar{\alpha} \text{ for } \alpha \in \bar{\Delta}(q)_+\}$$

where $\bar{\alpha} = (\cdot) + (\cdot)$ for $\alpha = (\cdot) \times (\cdot)$ and $\bar{\Delta}(q)_+ = \bar{\Delta}(q)_+ \setminus \{\infty\}$.

- (4) B_q is a subalgebra of $W(n)$.

Proof. The first three assertions are clear. In order to prove the last one, we only need to check the operator "+" is closed in $\bar{\Delta}(q)_+$ in view of the first three assertions, and Formula (3.2), along with Corollary 2.4. By a direct check, we easily know that it is closed in $\bar{\Delta}(0)_+$ and $\bar{\Delta}(n)_+$. As to an arbitrary q , we note that in

$$B_q = B_0(x_1, \dots, x_{n-q}) + Q_q + B_q(x_{n-q+1}, \dots, x_n)$$

the first and last summands are already known to be subalgebras. Furthermore, Q_q is normalized by the other summands. So we only need to prove Q_q is a subalgebra.

For this, it is sufficient to check that the operator " + " is closed in $\bar{\Delta}\{Q\}_+$. By a straightforward and easy computation, " + " is closed there. \square

3.3. Standard Borel subalgebras. In the concluding subsection, we prove all B_q are Borel subalgebras of $W(n)$.

Proposition 3.3. *Maintain the notations as before. All Borel subspaces B_r , $r = 1, 2, \dots, n$ are Borel subalgebras of $W(n)$, called the standard Borel subalgebras.*

Proof. By the definition of Borel subspaces, every Borel subspace contains the maximal torus \mathfrak{t}_0 of $W(n)$. According to Lemma 3.2(4), those Borel subspaces are subalgebras of $W(n)$. The remaining things are the following two, one is verification for the solvableness of Borel subspaces, and the other one is for the maximality.

(1) Claim: all Borel subspaces are solvable. We only need to prove that $B_q^{(m)} = 0$ for some positive integer m .

By Lemma 2.6, we know B_0 is a solvable subalgebra. As to B_n , we first observe that $B_n^{(n+1)} \subset B_0$, from which it follows that $B_n^{(n+1)}$ is solvable. Hence B_n itself is solvable.

Finally, let us check B_q . Recall $B_q = B_0(x_1, \dots, x_{n-q}) + Q_q + B_q(x_{n-q+1}, \dots, x_n)$. The first and last summand subalgebras as the naught-varied and full-varied Borel subspaces have been known solvable. Note that the middle summand subalgebra Q_q is normalized by the first and last one. So we only need to verify that Q_q is solvable. This follows that $Q_q^{(q)} \subset W(n)_1$. The claim is proved.

(2) Claim: all Borel spaces are not contained properly in any solvable subalgebras.

For $B_0 = \mathfrak{b} + W(n)_1$, the maximality is easily seen. Actually, if a solvable algebra $\mathfrak{H} \supseteq B_0$, then \mathfrak{H} is \mathfrak{t}_0 -graded by Corollary 2.4. Then $\mathfrak{H} = \sum_q \mathfrak{H}_{[q]}$ with $\mathfrak{H}_{[q]} = \mathfrak{H} \cap W(n)_{[q]}$. Note that the subalgebra generated by $\partial_i, x_i \partial_i, x_i^2 \partial_i$ is not solvable. So $\mathfrak{H}_{[-1]}$ must be zero. Hence $\mathfrak{H} \subset W(n)_0$. On the other hand, $\mathfrak{H}_{[0]}$ is solvable in $W(n)_{[0]}$. However $\mathfrak{b} \subset \mathfrak{H}_{[0]} \subset W(n)_{[0]}$, and \mathfrak{b} is already a Borel subalgebra of $W(n)_{[0]}$. Hence $\mathfrak{H}_{[0]}$ coincides with \mathfrak{b} . Thus $\mathfrak{H} = B_0$. The maximality of B_0 is true.

As to B_n , recall

$$B_n = W(n)_{[-1]} + \mathfrak{b} + \sum_{q=1}^n \sum_{\mathbf{a}(q)} \mathbb{K} \mathbf{x}^{\mathbf{a}(q)} \partial_q$$

where $\mathbf{a}(q) := (a_1, \dots, a_q) \in \mathbb{P}^q$ with $|\mathbf{a}(q)| > 1$, a_i runs over \mathbb{P} for $i = 1, \dots, q-1$, and $a_q = 0, 1$. Suppose there is a solvable algebra $\mathfrak{H} \supseteq B_n$. Similar to the above argument, we can have that $\mathfrak{H} = \sum_q \mathfrak{H}_{[q]}$ with $\mathfrak{H}_{[q]} = \mathfrak{H} \cap W(n)_{[q]} \supset (B_n)_{[q]}$ with $\mathfrak{H}_{[-1]} = W(n)_{[-1]}$ and $\mathfrak{H}_{[0]} = \mathfrak{b}$. If \mathfrak{H} contains properly B_n , then there must be a non-zero element $X \in \mathfrak{H}_\alpha \subset \mathfrak{H}_{[s]}$ for $s > 0$ which is not in $(B_n)_{[s]}$. We can write, with aid of the notation $\mathbf{x} = (x_1, \dots, x_n)$,

$$X = \sum_{q=1}^n \sum_{\mathbf{b}(q)} C_{\mathbf{b}(q)} \mathbf{x}^{\mathbf{b}(q)} \partial_q$$

where $C_{\mathbf{b}(q)} \in \mathbb{K}$, $\mathbf{b}(q) = (b_1, \dots, b_n) \in \mathbb{P}^n$ satisfying either $(b_{q+1}, \dots, b_n) \neq 0$, or $(b_{q+1}, \dots, b_n) = 0$ with $b_q > 1$, as long as $C_{\mathbf{b}(q)} \neq 0$. Say, either $b_t \neq 0$ for some $t > q$, or $b_q = d > 2$. Under a suitable sequence of adjoint actions $\text{ad} D$ for some $D \in \mathfrak{B}_{[-1]}$, one of the following situations happens: either \mathfrak{H} contains a subalgebra generated by $x_t \partial_q, x_q \partial_t, x_t \partial_t - x_q \partial_q$, or \mathfrak{H} contains a subalgebra generated by $x_q^2 \partial_q, x_q \partial_q, \partial_q$. But both kinds of the subalgebras are not solvable, which contradicts with the solvableness of \mathfrak{H} . Hence, \mathfrak{H} must coincide with \mathfrak{B}_n . We complete the proof of the maximality of \mathfrak{B}_n .

Let us finally investigate the maximality of general Borel subspaces

$$\mathfrak{B}_q = \mathfrak{B}_0(x_1, \dots, x_{n-q}) + Q_q + \mathfrak{B}_q(x_{n-q+1}, \dots, x_n).$$

We need to prove that for any solvable subalgebra \mathfrak{H} , $\mathfrak{H} \supseteq \mathfrak{B}_q$ implies $\mathfrak{H} = \mathfrak{B}_q$. Observe the containing relation of two solvable subalgebras in $W(x_1, \dots, x_{n-q})$ that $\mathfrak{H} \cap W(x_1, \dots, x_{n-q}) \supseteq \mathfrak{B}_0(x_1, \dots, x_{n-q})$, along with a result just proved that $\mathfrak{B}_0(x_1, \dots, x_{n-q})$ is a Borel subalgebra of $W(x_1, \dots, x_{n-q})$. It follows that $\mathfrak{H} \cap W(x_1, \dots, x_{n-q}) = \mathfrak{B}_0(x_1, \dots, x_{n-q})$. By the same reason, we have $\mathfrak{H} \cap W(x_{n-q+1}, \dots, x_n) = \mathfrak{B}_q(x_{n-q+1}, \dots, x_n)$. Thus we have

$$\mathfrak{B}_q \subseteq \mathfrak{H} \subseteq \mathfrak{B}_0(x_1, \dots, x_{n-q}) + \overline{Q_q} + \mathfrak{B}_q(x_{n-q+1}, \dots, x_n)$$

where $\overline{Q_q} = \sum_{i=1}^{n-r} \sum_{\mathbf{a}(i), \mathbf{b}(i)} \mathbb{K} \mathbf{u}^{\mathbf{a}(i)} \mathbf{w}^{\mathbf{b}(i)} \partial_i + \sum_{j=n-r+1}^n \sum_{\mathbf{a}(j), \mathbf{b}(j)} \mathbb{K} \mathbf{u}^{\mathbf{a}(j)} \mathbf{w}^{\mathbf{b}(j)} \partial_j$ with \mathbf{u} and \mathbf{w} being the same meaning as in (3.1), and $\mathbf{a}(-) := (a_1, \dots, a_{n-r})$ running over \mathbb{P}^{n-r} for $(-) = (i), (j)$, and $\mathbf{b}(-) := (b_{n-r+1}, \dots, b_n)$ running over \mathbb{P}^r for $(-) = (i), (j)$, subjected to the condition that $|\mathbf{a}(j)| > 0$ and $|\mathbf{b}(i)| > 0$.

Suppose $\mathfrak{H} \supsetneq \mathfrak{B}_q$. Then there must be some element $X \in \overline{Q_q} \setminus Q_q$. Comparing $\overline{Q_q}$ and Q_q , we have

$$X = \sum_{i=1}^{n-r} \sum_{\mathbf{a}(i), \mathbf{b}(i)} C_{\mathbf{a}(i), \mathbf{b}(i)} \mathbf{u}^{\mathbf{a}(i)} \mathbf{w}^{\mathbf{b}(i)} \partial_i \quad (3.3)$$

with $C_{\mathbf{a}(i), \mathbf{b}(i)} \in \mathbb{K}$, and with $\mathbf{a}(i), \mathbf{b}(i)$ violating the condition listed below (3.1) as long as $C_{\mathbf{a}(i), \mathbf{b}(i)} \neq 0$. Say, $|\mathbf{a}(i)| = 1 = a_j$ for some $1 \leq i < j \leq n - q$ and $C_{\mathbf{a}(i), \mathbf{b}(i)} \neq 0$. Thus \mathfrak{H} contains $x_j \partial_i, x_i \partial_j$ and $x_i \partial_j - x_j \partial_i$, thereby contains the subalgebra generated by them, which is not solvable. This contradicts with the hypothesis that \mathfrak{H} is solvable. Hence, it is false that $\mathfrak{H} \supsetneq \mathfrak{B}_q$. Thus, we complete the proof of the maximality of \mathfrak{B}_q . Summing up the two statements, we complete the proof. \square

4. CONJUGACY CLASSES OF BOREL SUBALGEBRAS

Maintain the notations and conventions as before. Especially, the characteristic p of the ground field \mathbb{K} is assumed to be bigger than 3 throughout this section.

4.1. In this section, we will prove that the standard Borel subalgebras are representatives of conjugacy classes of all Borel subalgebras. Let \mathcal{B} be any given Borel

subalgebra of $W(n)$. By the definition of Borel subalgebras, \mathcal{B} contains a maximal torus. We introduce a number $r(\mathcal{B})$ associated with the conjugacy class of \mathcal{B} , with

$$r(\mathcal{B}) := \max\{r \mid \text{there exists } \sigma \in \text{Aut}(W(n)) \text{ such that } \sigma(\mathfrak{t}_r) \subset \mathcal{B}\}.$$

The following conclusion is obvious.

Lemma 4.1. $r(\mathcal{B}_r) = r$.

4.2. Let us begin the arguments with two special cases $r(\mathcal{B}) = 0$ and $r(\mathcal{B}) = n$.

Lemma 4.2. *Assume \mathcal{B} is a Borel subalgebra of $W(n)$ with $r(\mathcal{B}) = 0$. Then $\mathcal{B} \cong \mathcal{B}_0$.*

Proof. Up to conjugation, we might as well assume $\mathcal{B} \supset \mathfrak{t}_0$. By Lemma 2.3, \mathcal{B} admits standard-graded structure. So $\mathcal{B} = \sum_i \mathcal{B}_{[i]}$. We claim that $\dim \mathcal{B}_{[-1]} = 0$. Actually, if there is a non-zero $D \in \mathcal{B}_{[-1]}$ which is expressed as $D = \sum_i c_{t_i} \partial_{t_i}$ with $c_{t_i} \in \mathbb{K} \setminus \{0\}$ for $t_i \in \{1, \dots, n\}$. Those ∂_{t_i} must fall in \mathcal{B} because $[x_{t_i} \partial_{t_i}, D] = c_{t_i} \partial_{t_i} \in \mathcal{B}$. Hence $(1 + x_{t_i}) \partial_{t_i} \in \mathcal{B}$, which contradicts with $r(\mathcal{B}) = 0$. Therefore, \mathcal{B} falls in $W(n)_0$. The remaining thing is done, thanks to Lemma 2.6. \square

4.3. Let \mathcal{B} be a Borel subalgebra of $W(n)$ with $r(\mathcal{B}) = n$, this is to say, $\mathcal{B} \supset \sigma(\mathfrak{t}_n)$ for $\mathfrak{t}_n = \mathbb{K}(1 + x_1) \partial_1 + \dots + \mathbb{K}(1 + x_n) \partial_n$, $\sigma \in \text{Aut}(W(n))$.

Lemma 4.3. \mathcal{B} is conjugate to \mathcal{B}_n .

Proof. We might as well assume $\mathcal{B} \supset \mathfrak{t}_n$. According to Corollary 2.4, \mathcal{B} must be \mathfrak{t}_n -graded. Assume $\dim \mathcal{B}_{[-1]}^{(\mathfrak{t}_n)} = r'$. Set $r := n - r'$. We will prove the statement by induction on r .

When $r = 0$, $n = r'$. and $\dim \mathcal{B}_{[-1]}^{(\mathfrak{t}_n)} = n$. Thus $\mathcal{B} \supset \mathfrak{t}_0$. By Lemma 2.3, $\mathcal{B} = \sum_i \mathcal{B}_{[i]}$, is standard-graded. Among the graded-homogeneous subspaces of \mathcal{B} , $\mathcal{B}_{[-1]} = W(n)_{[-1]}$ and $\mathcal{B}_{[0]} \supset \mathfrak{t}_0$. We claim that

(*) with some permutation $(s_1 \dots s_n)$ of $(1 2 \dots n)$, $\mathcal{B}_0 \subset \sum_i \sum_{\mathbf{a}(i)} \mathbb{K} \mathbf{x}^{\mathbf{a}(i)} \partial_{s_i}$ where $\mathbf{x} = (x_{s_1}, \dots, x_{s_n})$, and $\mathbf{a}(i) = (a_1, \dots, a_i, 0, \dots, 0) \in \mathbb{P}^n$ subjected to the condition that $a_i = 0$ or $a_i = 1$.

Actually, if the claim (*) is not true, by \mathfrak{t}_0 -grading property of \mathcal{B} and by the suitable composition of $\text{ad} \partial_s$ -action for $s = 1, \dots, n$ we conclude that one of the following two kinds of situations happens:

- (†) there exist in \mathcal{B} a pair of elements $x_i \partial_j$ and $x_j \partial_i$ with $i \neq j$.
- (‡) there exists $x_i^2 \partial_i$ in \mathcal{B} for some i .

The former means that there are a triple in \mathcal{B} : $x_i \partial_j$, $x_j \partial_i$ and $x_i \partial_i - x_j \partial_j$; the latter means that there are a triple in \mathcal{B} : ∂_i , $x_i \partial_i$ and $x_i^2 \partial_i$. Any one of such situations contradict with the solvableness of \mathcal{B} . Hence the claim (*) is true. Thus, by a suitable permutation τ of (x_1, \dots, x_n) which gives rise an automorphism of $W(n)$, \mathcal{B} is isomorphic to a solvable subalgebra of \mathcal{B}_n . Hence $\mathcal{B} \cong \mathcal{B}_n$ by the maximal solvableness of Borel subalgebras. We complete of the proof for $r = 0$.

In the remainder of the proof, we inductively hypothesize that $r > 0$, and that the statement has held for the case of being less than r . We might as well assume

$\mathcal{B}_{[-1]}^{(t_n)} = \mathbb{K}\partial_1 + \cdots + \mathbb{K}\partial_r$ without loss of generality. Then $\mathcal{B} \supset \mathfrak{t}_{n-r} = \mathbb{K}x_1\partial_1 + \cdots + \mathbb{K}x_r\partial_r + \mathbb{K}(1+x_{r+1})\partial_{r+1} + \cdots + \mathbb{K}(1+x_n)\partial_n$. Set $\mathcal{B}_1 := \mathcal{B} \cap W(x_1, \dots, x_r)$ and $\mathcal{B}_2 := \mathcal{B} \cap W(x_{r+1}, \dots, x_n)$. Then

$$\begin{aligned} \mathcal{B}_1 &\supset \sum_{i=1}^r \mathbb{K}\partial_i + \sum_{i=1}^r \mathbb{K}x_i\partial_i; \\ W(x_{r+1}, \dots, x_n)_0^{(t_{n-r})} &\supset \mathcal{B}_2 \supset \sum_{i=r+1}^n \mathbb{K}(1+x_i)\partial_i. \end{aligned} \quad (4.1)$$

The further arguments will be divided into two cases.

The first case: there is no nilpotent element in \mathcal{B}_2 . Then $\mathcal{B}_2 = \mathbb{K}(1+x_{r+1})\partial_{r+1} + \cdots + \mathbb{K}(1+x_n)\partial_n$. And

$$\mathcal{B} = \mathcal{B}_1 + \sum_{(b_1, \dots, b_r; c_{r+1}, \dots, c_n; i) \in I} \mathbb{K}x_1^{b_1} \cdots x_r^{b_r} (1+x_{r+1})^{c_{r+1}} \cdots (1+x_n)^{c_n} \partial_i + \mathcal{B}_2; \quad (4.2)$$

for a subset I of $\Gamma = \mathbb{P}^r \times \mathbb{P}^{n-r} \times \{1, \dots, r\}$ where $\mathbb{P} := \{0, 1, \dots, p-1\}$. We denote the second sum formula in (4.2) by \mathcal{B}'_1 , and we then have

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}'_1 + \mathcal{B}_2, \text{ with } \mathcal{B}'_1 \subset \mathcal{B}. \quad (4.3)$$

Set \bar{I} to be a subset of Γ which consists of all elements $(b_1, \dots, b_r; \gamma_{r+1}, \dots, \gamma_n; i)$ as long as there exists $(b_1, \dots, b_r; c_{r+1}, \dots, c_n; i) \in I$ with γ_i running over \mathbb{P} , $i = r+1, \dots, n$. We simply denote elements of \bar{I} by $(b_1, \dots, b_r; \infty, \dots, \infty; i)$, which implies the existence of some $(b_1, \dots, b_r; -, \dots, -; i) \in I$. Set

$$\bar{\mathcal{B}}'_1 = \sum_{(b_1, \dots, b_r; \infty, \dots, \infty; i) \in \bar{I}} \mathbb{K}x_1^{b_1} \cdots x_r^{b_r} (1+x_{r+1})^\infty \cdots (1+x_n)^\infty \partial_i.$$

And set

$$\bar{\mathcal{B}} = \mathcal{B}_1 + \bar{\mathcal{B}}'_1 + \mathcal{B}_2.$$

We claim that

$$\bar{\mathcal{B}} \text{ is a solvable subalgebra of } W(n) \text{ containing } \mathcal{B}, \quad (4.4)$$

thereby \mathcal{B} coincides with $\bar{\mathcal{B}}$ according to the maximality of \mathcal{B} . To prove the above claim, we first need to prove that the linear subspace $\bar{\mathcal{B}}$ is a subalgebra. For this, observe that $\mathcal{B}_1 + \bar{\mathcal{B}}'_1$ is normalized by \mathcal{B}_2 . We only need to prove $\mathcal{B}_1 + \bar{\mathcal{B}}'_1 = \bar{\mathcal{B}}'_1$ is a subalgebra. Actually, for any $\bar{X}, \bar{X}' \in \bar{\mathcal{B}}'_1$ with $\bar{X} = x_1^{b_1} \cdots x_r^{b_r} (1+x_{r+1})^\infty \cdots (1+x_n)^\infty \partial_i$, $\bar{X}' = x_1^{b'_1} \cdots x_r^{b'_r} (1+x_{r+1})^\infty \cdots (1+x_n)^\infty \partial_j$, we have $X = x_1^{b_1} \cdots x_r^{b_r} (1+x_{r+1})^{c_{r+1}} \cdots (1+x_n)^{c_n} \partial_i \in \mathcal{B}'_1$ and $X' = x_1^{b'_1} \cdots x_r^{b'_r} (1+x_{r+1})^{c'_{r+1}} \cdots (1+x_n)^{c'_n} \partial_j \in \mathcal{B}'_1$ for some $(c_{r+1}, \dots, c_n), (c'_{r+1}, \dots, c'_n) \in \mathbb{P}^{n-r}$. And then

$$[\bar{X}, \bar{X}'] = (1+x_{r+1})^\infty \cdots (1+x_n)^\infty [x_1^{b_1} \cdots x_r^{b_r} \partial_i, x_1^{b'_1} \cdots x_r^{b'_r} \partial_j] \in \bar{\mathcal{B}}'_1 \quad (4.5)$$

because $[X, X'] \in \mathcal{B}'_1$, and $\mathcal{B}'_1 \subset \mathcal{B}$ is \mathfrak{t}_{n-r} -graded. The solvability of $\overline{\mathcal{B}}$ follows from (4.5), along with the normalization of $\overline{\mathcal{B}}'_1$ to \mathcal{B}_1 , and with the solvability of \mathcal{B} . Hence, we complete the proof of $\mathcal{B} = \overline{\mathcal{B}}$. On the other hand, the structure of $\overline{\mathcal{B}}$ implies that $\mathcal{B} + \mathbb{K}\partial_n = \overline{\mathcal{B}} + \mathbb{K}\partial_n$ is a solvable subalgebra of $W(n)$, which contradicts with the maximal solvableness of \mathcal{B} because $\mathcal{B} + \mathbb{K}\partial_n \supsetneq \mathcal{B}$. So the present case couldn't happen.

The second case: there exists some nilpotent element X in \mathcal{B}_2 . Then $\mathcal{B}_2 \not\supseteq \mathfrak{t}_{n-r}(x_{r+1}, \dots, x_n) = \mathbb{K}(1 + x_{r+1})\partial_{r+1} + \dots + \mathbb{K}(1 + x_n)\partial_n$. In such a case, we can assume $X = (1 + x_{r+1})^{a_{r+1}} \dots (1 + x_n)^{a_n} \partial_q$ for a certain $q \in \{r+1, \dots, n\}$. The nilpotency of X implies that either $a_q = 0$ or $a_q \geq 2$. According to the argument in §2.2, we can define an automorphism φ of the truncated polynomial $A(n)$ via

$$\begin{aligned} \varphi : x_j &\mapsto x_j \text{ for } j \in \{1, \dots, n\} \setminus \{q\}, \\ \varphi : x_q &\mapsto (1 + x_q) \prod_{j \in \{r+1, \dots, n\}, j \neq q} (1 + x_j)^{d_j} - 1 \end{aligned} \quad (4.6)$$

where $d_j = d_q(p - a_j) \in \{0, 1, \dots, p-1\}$ with $d_q(a_q - 1) \equiv 1 \pmod{p}$. Then φ induces an automorphism $\overline{\varphi}$ of $W(n)$, denoted by Φ . The inverse of φ is easily presented as follows:

$$\begin{aligned} \varphi^{-1} : x_j &\mapsto x_j \text{ for } j \in \{1, \dots, n\} \setminus \{q\}, \\ \varphi^{-1} : x_q &\mapsto (1 + x_q) \prod_{j \in \{r+1, \dots, n\}, j \neq q} (1 + x_j)^{p-d_j} - 1. \end{aligned} \quad (4.7)$$

Then $\Phi|_{\mathcal{B}_1} = \text{identity}$. We have

$$\begin{aligned} \Phi(\partial_q) &= \varphi \circ \partial_q \circ \varphi^{-1} \\ &= \prod_{j \in \{r+1, \dots, n\}, j \neq q} (1 + x_j)^{p-d_j} \partial_q. \end{aligned} \quad (4.8)$$

And then we have

$$\begin{aligned} \Phi(X) &= \Phi((1 + x_{r+1})^{a_{r+1}} \dots (1 + x_n)^{a_n} \partial_q) \\ &= \Phi\left(\left(\prod_{j \in \{r+1, \dots, n\}, j \neq q} (1 + x_j)^{a_j}\right)(1 + x_q)^{a_q} \partial_q\right) \\ &= \varphi\left(\left(\prod_{j \in \{r+1, \dots, n\}, j \neq q} (1 + x_j)^{a_j}\right)(1 + x_q)^{a_q}\right) \Phi(\partial_q) \\ &= \left(\prod_{j \in \{r+1, \dots, n\}, j \neq q} (1 + x_j)^{a_j}\right) \varphi((1 + x_q)^{a_q}) \Phi(\partial_q) \\ &= \left(\prod_{j \in \{r+1, \dots, n\}, j \neq q} (1 + x_j)^{a_j}\right) (1 + x_q)^{a_q} \prod_{j \in \{r+1, \dots, n\}, j \neq q} (1 + x_j)^{d_j a_q} \\ &\quad \prod_{j \in \{r+1, \dots, n\}, j \neq q} (1 + x_j)^{p-d_j} \partial_q \\ &= (1 + x_q)^{a_q} \partial_q. \end{aligned} \quad (4.9)$$

Furthermore, $\Phi((1+x_j)\partial_j) = (1+x_j)\partial_j$ for $j \in \{r+1, \dots, n\} \setminus \{q\}$. And

$$\begin{aligned} \Phi((1+x_q)\partial_q) &= \varphi(1+x_q)\Phi(\partial_q) \\ &= (1+x_q) \prod_{j \in \{r+1, \dots, n\}, j \neq q} (1+x_j)^{d_j} \prod_{j \in \{r+1, \dots, n\}, j \neq q} (1+x_j)^{p-d_j} \partial_q \\ &= (1+x_q)\partial_q. \end{aligned}$$

Hence we have the following properties

- (Φ-1) $\Phi|_{\mathfrak{t}_{n-r}(x_{r+1}, \dots, x_n)} = \text{identity}$.
- (Φ-2) Under Φ , $\Phi(\mathcal{B}) = \mathcal{B}_1 + \Phi(\mathcal{B}_2)$ which contains \mathfrak{t}_n , and intersects with $W(n)_{[-1]}^{(n)}$ at $\sum_{i=1}^r \mathbb{K}\partial_i$.
- (Φ-3) It's a specially important consequence that $\Phi(\mathcal{B}_2)$ admits a nilpotent element $\Phi(X) = (1+x_q)^{a_q}\partial_q$.

Next, for $\Phi(\mathcal{B})$ we have an automorphism Ψ which arises from $\psi \in \text{Aut}(A(n))$, with

$$\psi(x_j) = \begin{cases} x_j, & \text{if } j \neq q; \\ (1+x_q)^{b_q} - 1, & \text{if } j = q; \end{cases}$$

where $b_q \in \{1, \dots, p-1\}$ with $(p-a_q+1)b_q \equiv 1 \pmod{p}$. Then we have the following items for $\Psi(\Phi(\mathcal{B}))$:

- (Ψ-1) $\Psi \circ \Phi|_{\mathfrak{t}_{n-r}(x_{r+1}, \dots, x_n)} = \text{identity}$.
- (Ψ-2) $\Psi(\Phi(\mathcal{B})) = \mathcal{B}_1 + \Psi(\Phi(\mathcal{B}_2))$ which contains \mathfrak{t}_n , and intersects with $W(n)_{[-1]}^{(t_n)}$ at $\sum_{i=1}^r \mathbb{K}\partial_i + \mathbb{K}\partial_q$.
- (Ψ-3) It's a specially important consequence that $\Psi(\Phi(\mathcal{B}_2))$ admits a nilpotent element ∂_q .

Thus, $\Psi \circ \Phi(\mathcal{B}) \supset \mathfrak{t}_n$ with $\dim \Psi \circ \Phi(\mathcal{B})_{[-1]}^{(t_n)} = r' + 1$. By the inductive hypothesis, we know that $\Psi \circ \Phi(\mathcal{B})$ is conjugate to \mathcal{B}_n . We complete the proof. \square

4.4. It's the position for us to investigate the general cases.

Lemma 4.4. *Let \mathcal{B} be any given Borel subalgebra of $W(n)$ and $r := r(\mathcal{B})$. Then \mathcal{B} is conjugate to \mathcal{B}_r .*

Proof. If $r = n$, then the lemma holds, thanks to Lemma 4.3. In the following argument, we assume $r < n$. Up to conjugation, we might as well assume \mathcal{B} contains $\mathfrak{t}_r = \mathbb{K}x_1\partial_1 + \mathbb{K}x_2\partial_2 + \dots + \mathbb{K}x_{n-r}\partial_{n-r} + \mathbb{K}(1+x_{n-r+1})\partial_{n-r+1} + \dots + \mathbb{K}(1+x_n)\partial_n$. Then \mathcal{B} is \mathfrak{t}_r -graded. According to the assumption, we have the following item:

$$(r-0) \mathcal{B}_{[-1]}^{(t_r)} \subset \mathbb{K}\partial_{n-r+1} + \dots + \mathbb{K}\partial_n.$$

Now $\mathcal{B}_{[0]}^{(t_r)} \cap W(x_1, \dots, x_{n-r})$ is a solvable subalgebra of $W(x_1, \dots, x_{n-r})_{[0]}$ containing the maximal torus $\mathfrak{t}_0(x_1, \dots, x_r)$. Note that $G_0 = \text{GL}(n, \mathbb{K}) \supset \text{GL}(n-r, \mathbb{K}) \times \text{GL}(r)$. By Proposition 1.4 we can further assume the following items, up to conjugation.

$$(r-1) \mathcal{B}_{[0]}^{(t_r)} \cap W(x_1, \dots, x_{n-r}) \subset \sum_{i \leq j; i, j=1, \dots, n-r} \mathbb{K}x_i\partial_j.$$

Set $\mathcal{B}_1 = \mathcal{B} \cap W(x_1, \dots, x_{n-r})$ which is solvable. Then $\mathfrak{t}_0(x_1, \dots, x_r) \subset \mathcal{B}_1 \subset W(r)_0$. According to Lemma 2.6, we may assume the following item:

(r-2) $\mathcal{B}_1 \subset$ the Borel subalgebra $\mathcal{B}_0(x_1, \dots, x_{n-r})$.

Set $\mathcal{B}_2 = \mathcal{B} \cap W(x_{n-r+1}, \dots, x_n)$. Then \mathcal{B}_2 contain the maximal torus

$$\mathfrak{t}_r(x_{n-r+1}, \dots, x_n)$$

of $W(x_{n-r+1}, \dots, x_n)$. Observe that

$$\text{Aut}(W(x_1, \dots, x_{n-r})) \times \text{Aut}(W(x_{n-r+1}, \dots, x_n)) \subset \text{Aut}(W(n)).$$

So we may assume simultaneously the following item in view of Lemma 4.3,

(r-3) $\mathcal{B}_2 \subset \Theta(\mathcal{B}_r(x_{n-r+1}, \dots, x_n))$. Here $\Theta \in \text{Aut}(W(x_{n-r+1}, \dots, x_n))$ arises from $\theta \in \text{Aut}(A(x_{n-r+1}, \dots, x_n))$ defined via $\theta(x_{n-r+i}) = (1+x_{n-r+i})^{p-1}-1$ for $i = 1, \dots, r$. This does make sense, according to the argument of §2.2.

Set $\mathbf{x} = (x_1, \dots, x_{n-q})$ and $\mathbf{y} = (y_{n-q+1}, \dots, y_n)$ for $y_i = 1 + x_i$. Consider a subspace Q exposed in the sense of Conventions 3.1:

$$Q = \sum_{i=1}^{n-r} \sum_{\mathbf{a}(i), \mathbf{b}(i)} \mathbb{K} \mathbf{x}^{\mathbf{a}(i)} \mathbf{y}^{\mathbf{b}(i)} \partial_i + \sum_{j=n-r+1}^n \sum_{\mathbf{a}(j); \mathbf{b}(j)} \mathbb{K} \mathbf{x}^{\mathbf{a}(j)} \mathbf{y}^{\mathbf{b}(j)} \partial_j,$$

where $\mathbf{a}(i) := (a_1, \dots, a_{n-r}) \in \mathbb{P}^{n-r}$ is subjected to the condition that either $|\mathbf{a}(i)| > 1$ or $|\mathbf{a}(i)| = 1 = a_1 + \dots + a_i$, while $\mathbf{a}(j) \in \mathbb{P}^{n-r}$ is subjected to the condition $|\mathbf{a}(j)| > 0$, and $\mathbf{b}(-) := (b_{n-r+1}, \dots, b_n)$ runs over \mathbb{P}^r for $(-) = (i), (j)$. It's not hard to see that the following items hold

(r-4) Q is a subalgebra, which normalizes both $\mathcal{B}_0(x_1, \dots, x_{n-r})$ and $\Theta(\mathcal{B}_r(x_{n-r+1}, \dots, x_n))$.

(r-5) $\Theta(\mathcal{B}_r) = \mathcal{B}_0(x_1, \dots, x_{n-r}) + Q + \Theta(\mathcal{B}_r(x_{n-r+1}, \dots, x_n))$. (Regard Θ naturally as an automorphism of $W(n)$).

From (r-2), (r-3) and (r-4), we know that $\mathcal{B}_Q := \mathcal{B}_1 + \mathcal{B}_2 + Q$ is a solvable subalgebra. We will finally show the following, up to conjugation

(r-6) $\mathcal{B}_Q \supset \mathcal{B}$.

In order to prove (r-6), we only need to prove that \mathcal{B} contains, under the sense of conjugation, neither elements of the forthcoming form (4.10), nor elements of the forthcoming form (4.11), presented as

$$Y = \mathbf{y}^{\mathbf{b}(q)} \partial_q \text{ with } q \in \{1, 2, \dots, n-r\}; \quad (4.10)$$

and

$$Z = x_m \mathbf{y}^{\mathbf{b}(q)} \partial_q \text{ with } m, q \in \{1, 2, \dots, n-r\}, m > q. \quad (4.11)$$

Suppose there exists such an element Y as in (4.10) in \mathcal{B} . Clearly, $|\mathbf{b}(q)| > 0$ by (r-0). For $Y = (1+x_{n-r+1})^{b_{n-r+1}} \dots (1+x_n)^{b_n} \partial_q$. According to the argument of §2.2, one can consider an automorphism Ω of $W(n)$ induced by $\omega \in \text{Aut}(A(n))$,

which is defined via

$$\begin{aligned}\omega : x_i &\mapsto x_i \text{ for } i \in \{1, \dots, n\} \setminus \{q\}, \\ \omega : x_q &\mapsto x_q \prod_{j \in \{r+1, \dots, n\}} (1 + x_j)^{b_j}\end{aligned}$$

Then $\Omega(Y) = \partial_q$, while $\Omega(x_q \partial_q) = x_q \partial_q$, and we have $\Omega(\mathcal{B}) \supset \sum_{i=1, i \neq q}^{n-r} \mathbb{K} x_i \partial_i + \mathbb{K}(1 + x_q) \partial_q + \sum_{i=n-r+1}^n \mathbb{K}(1 + x_i) \partial_i \cong \mathfrak{t}_{r+1}$, which contradicts with $r(\mathcal{B}) = r$. We complete the proof of that \mathcal{B} does not contain any elements of the form (4.10).

For (4.11), we first show the following statement

\mathcal{B} does not contain any pair of elements of the forms:

$$\begin{aligned}Z(m, q) &= x_m (1 + x_{n-r+1})^{b_{n-r+1}} \cdots (1 + x_n)^{b_n} \partial_q, \\ Z(q, m) &= x_q (1 + x_{n-r+1})^{c_{n-r+1}} \cdots (1 + x_n)^{c_n} \partial_m,\end{aligned}\tag{4.12}$$

with $m, q \in \{1, 2, \dots, n-r\}, m > q$. Suppose that there are such a pair $Z(m, q)$ and $Z(q, m)$ appearing in \mathcal{B} . For simplicity of our arguments below, we abbreviate $Z(m, q)$, $Z(q, m)$ to $\Pi^{\mathbf{b}} x_m \partial_q$, $\Pi^{\mathbf{c}} x_q \partial_m$ respectively, where $\Pi^{\mathbf{b}}$, $\Pi^{\mathbf{c}}$ denotes respectively the products $(1 + x_{n-r+1})^{b_{n-r+1}} \cdots (1 + x_n)^{b_n}$, and $(1 + x_{n-r+1})^{c_{n-r+1}} \cdots (1 + x_n)^{c_n}$. We further denote H_m for $x_m \partial_m$, and H_q for $x_q \partial_q$. Note that \mathcal{B} contains the subalgebra generated by $Z(m, q)$ and $Z(q, m)$, denoted by $L(m, q)$. We have the Lie products in $L(m, q)$ as below

$$\begin{aligned}[\Pi^{\mathbf{b}} x_m \partial_q, \Pi^{\mathbf{c}} x_q \partial_m] &= \Pi^{\mathbf{b}+\mathbf{c}} (H_m - H_q); \\ [\Pi^{\mathbf{b}+\mathbf{c}} (H_m - H_q), \Pi^{\mathbf{b}} x_m \partial_q] &= \Pi^{2\mathbf{b}+\mathbf{c}} (2x_m \partial_q); \\ [\Pi^{\mathbf{b}+\mathbf{c}} (H_m - H_q), \Pi^{\mathbf{c}} x_q \partial_m] &= \Pi^{\mathbf{b}+2\mathbf{c}} (-2x_q \partial_m).\end{aligned}\tag{4.13}$$

Note that all elements like $\Pi^{\mathbf{b}}$ are invertible in $A(x_{n-r+1}, \dots, x_n)$, thereby being invertible linear transformation of $W(n)$. Thus $L(m, q)$ is not solvable, which contradicts with the solvableness of \mathcal{B} . We complete the proof of (4.12). Note that if \mathcal{B} contains $x_q \partial_m$ as in (r-1), then the statement (4.12) already assures that \mathcal{B} does not contain any elements of the form (4.11). Thus, up to conjugation we can assure that \mathcal{B} does not contain any elements of the form (4.11) (if necessary, we change the order of the indeterminants x_i for $i = 1, \dots, n-r$, which gives rise some automorphism of $W(n)$). We have completed the proof of (r-6).

It follows from (r-2)-(r-6) that $\mathcal{B} \subset \mathcal{B}_Q \subset \Theta(\mathcal{B}_r)$. The maximality of \mathcal{B} implies $\mathcal{B} = \Theta(\mathcal{B}_r)$. We complete the proof. \square

We have a direct consequence

Corollary 4.5. *The following statements hold.*

- (1) *Any Borel subalgebra of $W(n)$ contains a maximal torus conjugate to \mathfrak{t}_0 .*
- (2) *For a Borel subalgebra \mathcal{B} of $W(n)$, take $\mathcal{B}^0 \in G \cdot \mathcal{B}$ with $\mathcal{B}^0 \supset \mathfrak{t}_0$. Set $d(\mathcal{B}) = \dim \mathcal{B}_{[-1]}^0$. Then $d(\mathcal{B}) = r(\mathcal{B})$.*

So, there are two equal invariants of Borel subalgebras: $d(\mathcal{B})$ and $r(\mathcal{B})$ under $\text{Aut}(W(n))$. Thus, those different standard Borel subalgebras \mathcal{B}_i are mutually non-isomorphic.

4.5. Main theorem. By Lemma 4.4, Corollary 4.5 and [2, Proposition 3.3] (note that in [2], the order of the parameters in the subscripts of the notations $\{\mathfrak{t}_r\}$ is inverted to the one here), along with Remark 2.5, we have

Theorem 4.6. *Assume that $\mathfrak{g} = W(n)$, $G = \text{Aut}(W(n))$, and that the characteristic p of the ground field \mathbb{K} is bigger than 3. Then the following statements hold.*

- (1) *There are $(n+1)$ conjugacy classes of Borel subalgebras of $W(n)$ under G . The standard Borel subalgebras \mathcal{B}_i , $i = 0, 1, \dots, n$ are representatives of $(n+1)$ conjugacy classes.*
- (2) *There are only one conjugacy class of generic Borel subalgebras, which is conjugate to \mathcal{B}_n .*

Remark 4.7. (1) Let $\mathfrak{g} = W(n)$ and G be the adjoint group of \mathfrak{g} , and $\mathcal{B} = \mathcal{B}_n$ we have known by a result of Premet in [18],

$$\mathcal{N}(\mathfrak{g}) = \overline{G \cdot \mathcal{N}(\mathcal{B})} \quad (4.14)$$

(what we expect is that this statement holds for any finite-dimensional Lie algebra \mathfrak{g} and its generic Borel subalgebra \mathcal{B} if such a Borel exists). Actually, there Premet gave a stronger result that the nilpotent cone coincides with the closure of the orbit $G \cdot \mathfrak{D}$ of a regular nilpotent element \mathfrak{D} . With generic Borel subalgebras, we can read something more from this viewpoint. That \mathfrak{D} falls in the generic Borel subalgebra \mathcal{B}_n , not falling in any non-generic Borel subalgebras.

(2) When taking non-generic Borel subalgebras of $\mathfrak{g} = W(n)$, we can see Proposition 4.14 may not hold ever. For example, take $\mathcal{B} = \mathcal{B}_0$, then the G -saturation $G \cdot \mathcal{B} \subset \mathfrak{g}_0$, the Zariski closure of $G \cdot \mathcal{B}$ is still contained in \mathfrak{g}_0 , not equal to \mathfrak{g} . It is not the case of the first assertion of the proposition. Furthermore, by [18, Lemma 4 and Lemma 5], $\dim G \cdot \mathcal{N}(\mathcal{B}) < np^n - n = \dim \mathcal{N}(\mathfrak{g})$. So $\overline{G \cdot \mathcal{N}(\mathcal{B})} \subsetneq \mathcal{N}(\mathfrak{g})$, which is not the case of the second assertion of the proposition.

(3) The dimensions of all Borel subalgebras of $W(n)$ are presented as below, via \mathcal{B}_r , $r = 0, 1, \dots, n$:

$$\begin{aligned} \dim \mathcal{B}_r &= \dim \mathcal{B}_0(x_1, \dots, x_{n-r}) + \dim Q_r + \dim \mathcal{B}_r(x_{n-r+1}, \dots, x_n) \\ &= ((n-r)p^{n-r} - \frac{(n-r)(n-r-1)}{2}) + ((n-r)(p^{n-r} - \frac{n-r-1}{2})(p^r - 1) + \\ &\quad r(p^{n-r} - 1)p^r) + \frac{2(p^r - 1)}{p-1} \\ &= np^n - \frac{(n-r)(n-r-1)}{2}p^r + \frac{2(p^r - 1)}{p-1} - rp^r. \end{aligned}$$

(4) In view of [9, Theorem D], it is reasonable to expect that all maximal solvable subalgebras of $W(n)$ are Borel subalgebras if and only if $p > n$. When $p \leq n$,

the situation of conjugacy of maximal solvable subalgebras would become very complicated.

5. SOLVABLE SUBGROUPS ASSOCIATED WITH STANDARD BOREL SUBALGEBRAS

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, and G the adjoint group of \mathfrak{g} . Recall we have introduced the solvable subgroups of G associated with Borel subalgebras of \mathfrak{g} . We will be specially interested in $\text{Stab}_G(\mathcal{B}_{\text{gen}})$ and $G/\text{Stab}_G(\mathcal{B}_{\text{gen}})$ for a generic Borel subalgebra \mathcal{B}_{gen} when \mathfrak{g} admits generic tori. In this section, we investigate the solvable groups associated with standard Borel algebras in the case $\mathfrak{g} = W(n)$.

5.1. For the standard Borel subalgebras \mathcal{B}_r of $W(n)$, denote $S_r = \text{Stab}_G(\mathcal{B}_r)$, $r = 0, 1, \dots, n$. Under the isomorphism between $G_0 = \text{Aut}(W(n))_0$ and $\text{GL}(n, \mathbb{K})$, we have a standard Borel subgroup B_0 of G_0 which corresponds to the one consisting of invertible upper-triangular matrices of $\text{GL}(n, \mathbb{K})$. We have the following general description of S_r .

Proposition 5.1. *Keep the notations appearing in Theorem 2.1. Let B_0 be the standard Borel subgroup of G_0 corresponding to the one consisting of invertible upper-triangular matrices of $\text{GL}(n, \mathbb{K})$. The following statements hold.*

- (1) *The solvable subgroup S_0 coincides with the Borel subgroup $B_0 \ltimes U$ of G .*
- (2) *For $r = 0, 1, \dots, n$, $S_r = B_0 \ltimes U_r$ is a connected subgroup of the Borel subgroup $B_0 \ltimes U$, where $U_r = U \cap \text{Stab}_G(\mathcal{B}_r)$.*

Proof. The first assertion is immediate. In (2), it is easily verified that $S_r = B_0 \ltimes U_r$. It remains to prove the connectedness of U_r . We identify B_0 with the standard Borel subgroup of $\text{GL}(n, \mathbb{K})$ consisting of invertible upper-triangular matrices, which has a maximal torus group T_0 consisting of invertible diagonal matrices. Obviously, T_0 normalizes U_r . Take $\tau(t) = \text{diag}(t, t, \dots, t) \in T_0$ for a given $t \in \mathbb{K} \setminus \{0\}$. Given a unipotent element $g \in U_r$, all $\tau(t)g\tau(t)^{-1}$ belong to the same connected component of U_r as g . Note that G can be identified with $\text{Aut}(A(n))$. Under this identification, we can easily see that $\lim_{t \rightarrow 0} \tau(t)g\tau(t)^{-1} = \text{id}_{A(n)}$. Hence U_r is connected. We complete the proof. \square

5.2. Set $\mathfrak{g} = W(n)$ and $G = \text{Aut}(W(n))$ described as in Theorem 2.1. We use the notations B_{gen} for $\text{Stab}_G(\mathcal{B}_n)$ instead of S_n .

Proposition 5.2. *In aid of identification between G and $\text{Aut}(A(n))$, B_{gen} can be regarded as a subgroup of the isotropy group of the following algebra flag*

$$\mathbb{K}[x_1, \dots, x_n] \supset \mathbb{K}[x_1, \dots, x_{n-1}] \supset \dots \supset \mathbb{K}[x_1]. \quad (5.1)$$

Furthermore, $B_{\text{gen}} = B_0 \ltimes U_n$, with

$$U_n = \{g \in U \mid (g - \text{id})(x_i) \in (\mathbb{K} + \mathbb{K}x_i)\mathbb{K}[x_1, \dots, x_{i-1}], i = 1, \dots, n\}.$$

Proof. It is clear that B_{gen} contains both B_0 and U_n , thereby contains $B \ltimes U_n$. Next we prove the inverse inclusion.

For any given $\sigma \in B_{\text{gen}}$, by Theorem 2.1 we can write $\sigma = \sigma_0 \circ \sigma_1$ with $\sigma_0 \in G_0$ and $\sigma_1 \in U$. So σ_0 stabilizes $\mathfrak{b} \subset \mathcal{B}_n$, which means $\sigma_0 \in B_0 \subset B_{\text{gen}}$. Thus $\sigma_1 \in B_{\text{gen}}$.

It remains to show $\sigma_1 \in U_n$. Suppose σ_1 is not in U_n . Then there exists some i such that $\sigma_1(x_i) = x_i + x_j f_1 + x_i^2 f_2 + f_3$ with $j > i$, $f_1 \in \mathbb{K}[x_1, \dots, x_n]$, $f_2 \in \mathbb{K}[x_1, \dots, \widehat{x_j}, \dots, x_n]$, $f_3 \in (\mathbb{K} + \mathbb{K}x_i)\mathbb{K}[x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_n]$, and f_1, f_2 are not all zero. Here the notation $\widehat{x_i}, \widehat{x_j}$ means omitting the indeterminants x_i, x_j respectively. Note that $\sigma_1 \in U$. By a straightforward computation, $\sigma_1^{-1}(x_i) = x_i + x_j g_1 + x_i^2 g_2 + g_3$, with $g_1, g_2, g_3 \in A(n)$ being subjected to the same conditions as the one for f_1, f_2 and f_3 . Then $\bar{\sigma}_1(x_i \partial_i) = x_i \partial_i + (x_j f_1 + x_i^2 f_2 + f_3) \partial_i + *$ does not fall in B_n . This is a contradiction. Hence σ falls in $B_0 \ltimes U_n$. We complete the proof. \square

From the proposition, B_{gen} may not be a Borel subgroup of G unless $n = 1$. Thus G/B_{gen} is generally not a projective variety (cf. [11, Corollary 21.3.B]).

6. FUTURE TOPICS

6.1. Questions.

Question 6.1. Suppose $(\mathfrak{g}, [p])$ is a restricted Lie algebra with the adjoint group $G = \text{Aut}_p(\mathfrak{g})^\circ$.

- (1) Assume \mathcal{B}_{gen} is a generic Borel subalgebra, and $B_{\text{gen}} = \text{Stab}_G(\mathcal{B}_{\text{gen}})$. What can we say about G/B_{gen} ?
- (2) What about the geometry on $G \cdot \mathcal{B}_{\text{gen}}$ for $\mathfrak{g} = W(n)$ and $G = \text{Aut}(W(n))$?
- (3) What can we say about the connection between G/B_{gen} and representations of $W(n)$? Can we exploit some clue of geometric representations for $W(n)$ as in the classical case (cf. [1], [13])?

6.2. Remarks. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, G its adjoint group. Assume \mathfrak{g} admits generic tori and then generic Cartan subalgebras. Recall that a Borel subalgebra is called generic if it contains a generic Cartan subalgebra.

- (1) If $\mathcal{N}(\mathfrak{g})$ is the closure of an orbit of D , then we could expect D falls in a generic Borel subalgebra. Conversely, if \mathcal{B} is a generic Borel subalgebra, when does it hold that there exists a nilpotent element \mathfrak{D} in \mathcal{B} such that $\mathcal{N}(\mathfrak{g}) = \overline{G \cdot \mathfrak{D}}$?
- (2) Under the circumstance when \mathfrak{g} has generic elements, $\mathcal{N}(\mathfrak{g})$ should be irreducible, which is a conjecture of Premet (cf. [16], [17], [18], [22]).

If there is no generic elements in \mathfrak{g} , there could be an infinite number of maximal tori in \mathfrak{g} , then there could be an infinite number of Borel subalgebras. Under such a circumstance, it could happen that $\mathcal{N}(\mathfrak{g})$ is not irreducible, in connection with Corollary 1.8.

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